MECHANICS AND HEAT

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MECHANICS AND HEAT

A TEXT BOOK FOR COLLEGES
AND TECHNICAL SCHOOLS

BY

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PREFACE.

The authors have had a very definite point of view in the preparation of this elementary treatise on Mechanics and Heat, and they assume that those to whom this preface is addressed have read their introductory chapter, especially the section entitled The Science of Physics.

The most important function of the teacher of physics is to build the logical and mechanical structure of the science; the logical structure mainly by lecture and recitation work including a great deal of practice by the student in numerical calculation, and the mechanical structure by laboratory work. These two phases of physics study should run along together. This text, however, is intended as a basis for the work of the class-room.

In their experience, the authors have come to recognize four chief difficulties in the teaching of elementary physics, as follows:

One difficulty is that the native sense of most men is incapable, without stimulation and direction, of supplying the material upon which the logical structure of the science is intended to operate. It is necessary to direct the student's attention to many familiar things and to present in the class-room a great many experimental demonstrations of presumably familiar phenomena.

A second difficulty is that the human mind, intuitively habituated as it is to consider the important practical affairs of life, can hardly be turned to that minute consideration of apparently insignificant details which is so necessary in the scientific handling even of the most distinctly practical problems.

A third difficulty, which indeed runs through the entire front-of-progress of the human understanding, is that mind-stuff, which has been developed as correlative to certain aspects of our ancestral environment, must be rehabilitated in entirely new relations in fitting a man for the conditions of civilized life. Every teacher knows how much coercion is required for so little of this rehabili-

tation; but the bare possibility of the process is a remarkable fact, and that it is possible to the extent of bringing a Newton or a Pasteur out of a hunting and fishing ancestry is indeed wonderful.

A fourth difficulty is that the possibility of this rehabilitation of mind-stuff has grown up as a human faculty almost solely on the basis of language, and the essence of this rehabilitation is the formation of what we call ideas; whereas a great deal of our knowledge of physics is correlated in mechanisms. Familiarity with machinery is perhaps the most important condition for the successful study of the physical sciences, and the mechanisms of manipulation and measurement are essential parts of the structure of the physical sciences without which the study of physics degenerates into an ineffective exercise in meaningless speculation.

Everyone is familiar with the life history of the butterfly, how it lives first as a caterpillar and then undergoes a complete transformation into a winged insect. It is of course evident that the bodily organs of the caterpillar are not at all suited to the needs of a butterfly, the very food (of those species which take food) being entirely different. As a matter of fact, almost every portion of the bodily structure of the caterpillar is dissolved as it were into a formless pulp at the beginning of the transformation, and the organization of a flying insect then grows out from a central nucleus very much as a chicken grows in the food-stuff of an egg. So it is in the growth of a man. In early childhood, if the individual is favored by Fortune, he exercises and develops more or less extensively the primitive instincts and modes of the race in a free outdoor life, and the experiences so gained are so much mind-stuff to be dissolved and transformed under more or less coercion and constraint into an effective mind of the twentieth-century type.

Many of our teachers, especially those who handle the mathematical sciences, seem to think that ideas can be built up in young men's minds by a sort of hocus pocus, out of nothing;

but it seems to the authors that ideas like everything else in this world must be made out of something. All elemental knowledge, such as the knowing how to throw a ball, how to ride a bicycle, how to swim, or how to use a tool, seems to be locked in the marginal regions of the mind as a very substantial but very highly specialized kind of intuition, and the problem of teaching elementary physical science is in part the problem of how, by suggestion or otherwise, to drag this material into the field of consciousness where it may be transformed into a generalized logical structure. A formal and abstract presentation of the principles of elementary science tends more than anything else to inhibit the influx of this elemental knowledge from the marginal regions of the mind into the field of consciousness and results in the building up of a theoretical structure which can have no traffic with any mental field beyond its own narrow boundaries. Such a state of mind is but a kind of idiocy, and to call it a knowledge of science is silly scholasticism.

The best way to meet this quadruply difficult situation is to relate the teaching of the physical sciences as much as possible to the immediately practical things of life, and to go in for suggestiveness as the only way to avoid a total inhibition of the sense that is born with our students. It is *not*, however, a question of exactitude versus suggestiveness, for, indeed, both are necessary; exactitude relates chiefly to the realm of ideas whereas suggestiveness relates chiefly to the realms of objective reality. That is to say, suggestiveness is the one form of appeal to the rudimentary and elemental things of the mind which are more directly connected with objective reality than are the more highly abstracted ideas which operate almost wholly within the field of consciousness.

Such a method is certainly calculated to limber up our theories and put tham all at work, the pragmatic* method our friends the philosophers call it, a method which pretends to a conquering

*From the Greek word πραγμα, meaning action, from which our words *practice* and *practical* come.

destiny.* Whatever one may think of that philosophy of life which exhibits itself in a temperament of the most intensely practical and matter-to-fact type, it is certain that pragmatism is the only philosophy a science teacher can entertain and escape what to him is the most dangerous form of idolatry, science for its own sake.

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BARRY MACNUTT.

SOUTH BETHLEHEM, PA.,
June 22, 1910.

* See the interesting little book on *Pragmatism*, by William James, Henry Holt & Co.

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ELEMENTS OF MECHANICS.

INTRODUCTION.

Everyone knows the capability of the Indian for long-continued and serious effort in his primitive mode of life, and yet it is difficult to persuade an Indian "farmer" to plow. Everyone knows, also, that the typical college student is not stupid, and yet it is difficult to persuade the young men of practical and business ideals in our colleges and technical schools to study the abstract elements of science. Indeed, it is as difficult to get these young men to hold abstract things in mind as it is to get a young Indian to plow, and for almost exactly the same reason. The scientific handling of any problem requires a minute and painstaking consideration of details which are in themselves devoid of human value; and this quality of detachment, which pervades every branch of science, is the most serious obstacle to young people in their study of the sciences. But analysis is necessary, and it is the object of this introduction to convey to the student some understanding of this fact.

There is perhaps a narrowing tendency in the necessarily specialized courses of study of engineering and professional students, and the correction for this tendency would seem to lie outside of science study. However this may be, it is of the greatest importance that every student should realize two things in connection with his study of the physical sciences. The first is that the study of the physical sciences is exacting beyond all compromise, involving as it does a degree of coercion and constraint which it is beyond the power of any teacher greatly to mitigate; and second is that the completest science stands abashed before the infinitely complicated and fluid array of

phenomena of the material world, except only in the assurance which its method gives.

There is a kind of salamander, the axolotl, which lives a tadpole-like youth and never changes to the adult form unless a stress of dry weather annihilates his watery world; but he lives always and reproduces his kind as a tadpole, and a very odd-looking tadpole he is, with his lungs hanging from the sides of his head in feathery tassels, brownish in tint from the red blood inside and the mud that settles on the outside of the tiny streamers. When the aquatic home of the axolotl dries up, he quickly develops a pair of internal lungs, lops off his tassels and embarks on a new mode of life on land. If our young men are to develop beyond the tadpole stage they must meet with quick and responsive inward growth, that new and increasing "stress of dryness" as many are wont to call our modern age of science and industry. The purely impersonal character of science study represents a profound change in the things a young man is called upon to think about and in the character of his thinking, but the popular conception of the specialist as a "Mr. Dry-as-dust" is a caricature, for the dryness of science is by no means the same thing as the uninteresting quality of boredom.

The Laws of Motion.

The most prominent aspect of all phenomena is motion. In that realm of nature which is not of man's devising* motion is universal. In the other realm of nature, the realm of things devised, motion is no less prominent. Every purpose of our practical life is accomplished by movements of the body and by directed movements of tools and mechanisms.

The *laws* of motion. Every one has a sense of the absurdity

* Science as young people study it has two chief aspects, or in other words, it may be roughly divided into two parts, namely, the study of *the things which come upon us*, as it were, and the study of *the things which we deliberately devise.* The things that come upon us include weather phenomena and every aspect and phase of the natural world, the things we cannot escape; and the things we devise relate chiefly to the serious work of the world, the things we laboriously build and the things we deliberately and patiently seek.

of the idea of reducing the more complicated phenomena of nature to an orderly system of mechanical law. For to speak of motion is to call to mind first of all the phenomena that are associated with the excessively complicated, incessantly changing, turbulent and tumbling motion of wind and water. These phenomena have always had the most insistent appeal to us, they have confronted us everywhere and always, and life is an unending contest with their fortuitous diversity. The laws of motion! Let us consider the impetuous complexity of a great storm or the dreadful confusion of a railway wreck and understand that what we call the laws of motion, although they have a great deal to do with the ways in which we think, have very little to do with the phenomena of nature. The laws of motion! Truly the science of mechanics is circumscribed in utter repudiation of such universal phrase, and yet the ideas which constitute the laws of motion have an almost unlimited extent of legitimate range, and *these ideas must be possessed with a perfect precision if one is to acquire any solid knowledge whatever of the phenomena of motion.* The necessity of precise ideas. Herein lies the impossibility of compromise and the necessity of coercion and constraint; one must think so and so, there is no other way. Indeed there is always a conflict in the mind of even the most willing student due to the constraint which precise ideas place upon our vivid and primitively adequate sense of physical things. This conflict is perennial and it is by no means a one-sided conflict between mere crudity and refinement, for refinement ignores many things. Indeed, precise ideas not only help to form* our sense of the world in which we live but they inhibit sense as well and their rigid and unchallenged rule would be indeed a stress of dryness.

The laws of motion. We return again and yet again to the subject, for one is not to be deterred therefrom by any concession of inadequacy, no, nor by any degree of respect for the vivid youthful sense of those things which to suit our narrow purpose

* See discussion of Bacon's New Engine on page 14.

must be stripped completely bare. It is unfortunate, however, that the most familiar type of motion, the flowing of water and the blowing of the wind, is bewilderingly useless as a basis for the establishment of the simple and precise ideas which are called the "laws of motion," and which are the most important of the fundamental principles of physics. These ideas have in fact grown out of the study of the simple phenomena which are associated with the motion of bodies in bulk without perceptible change of form, the motion of rigid bodies, so-called.

Before narrowing down the scope of the discussion, however, let us illustrate a very general application of the simplest idea of motion, the idea of velocity. Every one has, no doubt, an idea of what is meant by the velocity of the wind; and a sailor, having what he calls a ten-knot wind, knows that he can manage his boat with a certain spread of canvas and that he can accomplish a certain portion of his voyage in a given time; but an experienced sailor, although he speaks glibly of a ten-knot wind, belies his speech by taking wise precaution against every conceivable emergency. He knows that a ten-knot wind is by no means a sure or a simple thing with its incessant blasts and whirls; and a sensitive anemometer, having more regard for minutiæ than any sailor, usually registers in every wind a number of almost complete but excessively irregular stops and starts every minute and variations of direction that sweep around half the horizon!

We must evidently direct our attention to something simpler than the wind. Let us, therefore, consider the drawing of a wagon or the propulsion of a boat. It is a familiar experience that effort is required to start a body moving and that continued effort is required to maintain the motion. Certain very simple facts as to the nature of this effort, as to the amount of effort required to produce motion and as to the conditions which determine the amount of effort required to keep a body in motion, were discovered by Sir Isaac Newton, and on the basis of these facts Newton formulated the "laws of motion."

The effort required to start a body or to keep it moving is called force. Thus, if one starts a box sliding along a table one is said to exert a force on the box. The same effect might be accomplished by interposing a stick between the hand and the box, in which case one would exert a force on the stick and the stick in its turn would exert a force on the box. We thus arrive at the notion of force action between inanimate bodies, between the stick and the box in this case, and Newton pointed out that the force action between the two bodies A and B always consists of two equal and opposite forces, that is to say, if body A exerts a force on B, then B exerts an equal and opposite force on A, or, to use Newton's words, **action is equal to reaction and in a contrary direction.**

In leading up to this statement one might consider the force with which a person pushes on the box and the equal and opposite force with which the box pushes back on the person, but if one does not wish to introduce the stick as an intermediary. it is better to speak of the force with which the hand pushes on the box, and the equal and opposite force with which the box pushes back on the hand, because in discussing physical phenomena it is of the utmost importance to pay attention to impersonal things. Indeed our modern industrial life, in bringing men face to face with an entirely unprecedented array of intricate mechanical and physical problems, demands of every one a great and increasing amount of impersonal thinking, and the precise and rigorous modes of thought of the physical sciences are being forced upon widening circles of men with a relentless insistance —all of which it was intended to imply by referring to the stress of dryness which overtakes the little axolotl in his contented existence as a tadpole.

When we examine into the conditions under which a body starts to move and the conditions under which a body once started is kept in motion, we come across a very remarkable fact, if we are careful to consider every force which acts upon the body, and this remarkable fact is that the forces which act upon a

body which remains at rest are related to each other in precisely the same way as the forces which act upon a body which continues to move steadily along a straight path. Therefore, it is convenient to consider, first the relation between the forces which act upon a body at rest, or upon a body in uniform motion, and then to consider the relation between the forces which act upon a body which is starting or stopping or changing the direction of its motion.

Suppose a person *A* were to hold a box in mid-air. To do so it would of course be necessary for him to push up on the box so as to balance the downward pull of the earth, the weight of the box, as it is called. Then if another person *B* were to take hold of the box and pull upon it in any direction, *A* would have to exert an equal pull on the box in the opposite direction to keep it stationary. That is to say, the forces which act upon a stationary body are always balanced.

Every one, perhaps, realizes that what is here said about the balanced relation of the forces which act upon a stationary box, is equally true of the forces which act on a box similarly held in a steadily moving railway car or boat. Therefore, the forces which act upon a body which moves steadily along a straight path are balanced.

This is evidently true when the moving body is surrounded on all sides by things which are moving along with it, as in a car or a boat; but how about a body which moves steadily in a straight path but which is surrounded by bodies which do not move along with it? Everyone knows that some active agent such as a horse or a steam engine must pull steadily upon such a body to keep it in motion, and that if left to itself such a moving body quickly comes to rest. Many have, no doubt, reached this further inference from experience, namely, that the tendency of moving bodies to come to rest is due to the dragging forces, or friction, with which surrounding bodies act upon a body in motion. Thus a moving boat is brought to rest by the drag of the water when the propelling force ceases to act; a train of cars

is brought to rest because of the drag due to friction when the pull of the locomotive ceases; a box which is moved across a table comes to rest when left to itself, because of the drag due to friction between the box and the table.

We must, therefore, always consider two distinct forces when we are concerned with a body which is kept in motion, namely, the propelling force due to some active agent such as a horse or an engine, and the dragging force due to surrounding bodies. Newton pointed out that when a body is moving steadily along a straight path, the propelling force is always equal and opposite to the dragging force. Therefore, **The forces which act upon a body which is stationary, or which is moving uniformly along a straight path, are balanced forces.**

Many hesitate to accept as a fact the complete and exact balance of propelling and dragging forces on a body which is moving steadily along a straight path in the open, but direct experiment shows it to be true, and the most elaborate calculations and inferences based upon this notion of the complete balance of propelling and dragging forces on a body in uniform motion are verified by experiment. One may ask, why a canal boat, for example, should continue to move if the pull of the mule does not exceed the drag of the water; but why should it stop if the drag does not exceed the pull? Understand that we are not considering the starting of the boat. The fact is that the conscious effort which one must exert to drive a mule, the cost of the mule, and the expense of his keep, are what most people think of, however hard one tries to direct their attention solely to the state of tension in the rope that hitches the mule to the boat after the boat is in full motion; and most people consider that if the function of the mule is simply to balance the drag of the water so as to keep the boat from stopping, then why should there not be some way to avoid the cost of so insignificant an operation? There is, indeed, an extremely important matter involved here which we will consider when we come to the discussion of work and energy; but it has no bearing on the matter of the

balance of propulsion and drag on a body which moves steadily along a straight path.

Let us now consider the relation between the forces which act upon a body which is changing its speed, upon a body which is being started or stopped, for example. Everyone has noticed how a mule strains at his rope when starting a canal boat, especially if the boat is heavily loaded, and how the boat continues to move for a long time after the mule ceases to pull. In the first case, the pull of the mule greatly exceeds the drag of the water, and the speed of the boat increases; and in the second case, the drag of the water of course exceeds the pull of the mule, for the mule is not pulling at all, and the speed of the boat decreases. When the speed of a body is changing, the forces which act on the body are unbalanced, and we may conclude that *the effect of an unbalanced force acting on a body is to change the velocity of the body;* and it is evident that the longer the unbalanced force continues to act the greater the change of velocity. Thus if the mule ceases to pull on a canal boat for one second the velocity of the boat will be but slightly reduced by the unbalanced drag of the water, whereas if the mule ceases to pull for two seconds the decrease of velocity will be much greater. *In fact the change of velocity due to a given unbalanced force is proportional to the time that the force continues to act.* This is exemplified by a body falling under the action of the unbalanced pull of the earth; after one second it will have gained a certain amount of velocity (about 32 feet per second), after two seconds it will have made a total gain of twice as much velocity (about 64 feet per second), and so on. Furthermore, since the velocity produced by an unbalanced force is proportional to the time that the force continues to act, it is evident that the effect of the force should be specified as *so-much-velocity-produced-per-second*, exactly as in the case of earning money, the amount one earns is proportional to the length of time that one continues to work, and we always specify one's earning capacity as so-much-money-earned-per-day.*

* See footnote on page 37.

Everyone knows what it means to give an easy pull or a hard pull on a body. That is to say, we all have the ideas of greater and less as applied to forces. Everybody knows also that if a mule pulls hard on a canal boat, the boat will get under way more quickly than if the pull is easy, that is, the boat will gain more velocity per unit of time under the action of a hard pull than under the action of an easy pull. Therefore, any precise statement of the effect of an unbalanced force on a given body must correlate the precise value of the force and the exact amount of velocity produced per unit of time by the force. This seems a very difficult thing, but its apparent difficulty is very largely due to the fact that as yet we have not agreed as to what we are to understand by the statement that one force is precisely three, or four, or any number of times as great as another. Suppose, therefore, that *we agree to call one force twice as large as another when it will produce in a given body twice as much velocity in a given time* (remembering of course that we are now talking about unbalanced forces, or that we are assuming for the sake of simplicity of statement, that no dragging forces exist). As a result of this definition we may state that the amount of velocity produced per second in a given body by an unbalanced force is proportional to the force.

Of course we know no more about the matter in hand than we did before we adopted the definition, but we do have a good illustration of how important a part is played in the study of science, by what we may call making up one's mind, in the sense of putting one's mind in order. This kind of thing is very prominent in the study of elementary physics, and that rather indefinite reference, in the story of the little tasseled tadpole, to an inward growth so needful before one can hope for any measure of success in our modern world of scientific industry was an allusion to this thing, the "making-up" of one's mind. Nothing is so essential in the acquirement of exact and solid knowledge as the possession of precise ideas, not indeed that a perfect precision is necessary as a means for retaining knowledge, but *that nothing*

*else so effectually opens the mind for the perception even of the simplest evidences of a subject.**

We have now settled the question as to the effect of different unbalanced forces on a given body on the basis of a very general experience, and by an agreement as to the precise meaning to be attached to the statement that one force is so many times as great as another; but how about the effect of the *same force* upon different bodies, and how may we identify the force so as to be sure that it *is* the same? It is required, for example, to exert a given force on body A and then to exert the same force on another body B. This can be done by causing a third body C (a coiled spring, for example) to exert the force; then the forces exerted on A and B are the same if the *reaction* in each case produces the same effect on body C (the same degree of stretch, for example). Concerning the effects of the same unbalanced force on different bodies three things have to be settled by experiment as follows:

(*a*) In the first place let us suppose that a certain force F is twice as large as a certain other force G, according to our agreement, because the force F produces twice as much velocity every second as force G when the one and then the other of these forces is caused to act upon a given body, a piece of lead for example. Then, does the force F produce twice as much velocity every second as the force G whatever the nature and size of the given body, whether it be wood, or ice, or sugar? Experiment shows that it does.

(*b*) In the second place, suppose that we have such amounts of lead, of iron, of wood, etc., that a certain given force produces the same amount of velocity per second when it is made to act, as an unbalanced force, upon one or another of these various bodies. Then what is the relation between the amounts of these various substances? Experiment shows that they all have the same mass in grams, or pounds, as determined by a balance.

* Opens the mind, that is, for those things which are conformable to or consistent with the ideas. The history of science presents many cases where accepted ideas have closed the mind to contrary evidences for many generations. Let young men beware!

That is, a given force produces the same amount of velocity per second in a given number of grams of any kind of substance. Thus the earth pulls with a certain definite force (in a given locality) upon M grams of any substance and, aside from the dragging forces due to air friction, all kinds of bodies gain the same amount of velocity per second when they fall under action of the unbalanced pull of the earth.

(*c*) In the third place, what is the relation between the velocity per second produced by a given force and the mass in grams (or pounds) of the body upon which it acts? Experiment shows that the velocity per second produced by a given force is inversely proportional to the mass of the body upon which the force acts. In speaking of the mass of a body in grams (or pounds) we here refer to *the result which is obtained by weighing the body on a balance scale*, and the experimental fact which is here referred to constitutes a very important discovery: namely, when one body has twice the mass of another, according to the balance method of measuring mass, it is accelerated half as fast by a given unbalanced force.

The effect of an unbalanced force in producing velocity may therefore be summed up as follows: **The velocity per second produced by an unbalanced force is proportional to the force and inversely proportional to the mass of the body upon which the force acts.** ***Furthermore,*** **the velocity produced by an unbalanced force is always in the direction of the force.**

Physical Measurement.

Among primitive races all things subject to exchange or barter are estimated by simple counting. Thus a Tartar herdsman estimates his wealth by counting his cattle. With the growth of civilization, however, there has been a great increase in the variety of useful and exchangeable commodities, and many of these commodities cannot be estimated by simple counting. The result has been that the simple operation of counting, which, of course, can be applied only to groups of separate and distinct

things, has developed into the operation called measurement, in which a continuous whole is estimated numerically by dividing it into equal, unit parts, and counting these parts. Thus oil or wine is counted out by means of a gallon measure, and cloth is counted out by means of a yard-stick.

In many kinds of measurement, the two distinct operations, (*a*) *dividing into equal unit parts* and (*b*) *counting* are obscured by the use of more or less elaborate measuring devices, but every measurement of whatever kind does, in fact, consist of these two fundamental operations. Thus in measuring a length by means of a *scale of inches*, the operation of dividing into unit parts has been performed once for all by the maker of the scale, and in this case the operation of counting is, in large part, "ready-made" by the numbers stamped on the scale. In the weighing of a consignment of coal, the operation of dividing into unit parts has been performed once for all by the maker of the *set of weights* and of the *divided balance beam*, and the operation of counting is, in large part, "ready-made" by the numbers stamped upon the weights and upon the beam.

The long experience of the race in estimating by the simple counting of separate things has given rise to a sense of sharp distinction between any two numbers, thus 1000 horses is clearly not the same thing as 999 horses; *but this sharp distinction between approximately equal numbers is devoid of physical significance in the case of numbers derived by measurement*, **because of the approximate character of the operation of dividing a whole into unit parts.** A person might buy a herd of horses supposing the number to be 1000, whereas a correct count would show 999; and although the purchaser might reasonably say, "Oh, let it go; it makes no difference," still the fact would remain that 999 horses is not 1000 horses; suppose, however, a man were to buy 1000 yards of cloth, he might remeasure the cloth and count 999 yards, but in remeasuring, the day may have been damp, or he may not have stretched the cloth in the same way as the manufacturer, or he may have taken more or less pains in fitting the

yard-stick to the successive portions of the cloth, or his yard-stick may have been in error. It is in fact impossible* to show that 1000 yards of cloth *is not* 999 yards of cloth, except by reasoning that 1000 pieces of silver is not 999 pieces of silver. A yard of cloth is not a separate thing whereas a piece of silver is.

The operation of dividing a length or an angle into equal unit parts for the purpose of measurement is an operation of *fitting* a standard to each part, an operation of congruence; and the actual measurement of any physical whole — let us not speak of it as a quantity until we have attached a number to it — depends upon one or another variety of congruence as a basis for the assumption of equality of the parts which are to be counted. Thus a pendulum may be assumed to mark off equal intervals of time because each movement of the pendulum is like the one that follows; and the equal arm balance is a device for indicating a certain kind of congruence between the body which is being weighed and the combination of weights which balances it.

The fundamental meaning of a physical quantity originates in and is defined by the actual operation of measuring that quantity. Thus, the mass of a body, as a quantity, is defined by the operation of weighing by a balance; and, since the result of this operation is always the same, within the limits of error, for a given amount of any substance, it is permissible to use this result as a measure of the amount of the substance. *Nearly every physical definition, rightly understood, is an actual physical operation.*

THE SCIENCE OF PHYSICS.

"We advise all men" says Bacon "to think of the true ends of knowledge, and that they endeavor not after it for curiosity, contention, or the sake of despising others, nor yet for reputation

*The principles involved in the measurement of cloth do not differ in any respect from the principles involved in more precise measurements in the laboratory. Every student of physics should be to some extent familiar with the theory of errors of observation. Among the best books on this subject are Holman's *Precision of Measurements* (Wiley & Sons) and Merriman's *Least Squares* (Wiley & Sons).

or power or any other such inferior consideration, but solely for the occasions and uses of life." It is difficult to imagine any other basis upon which the study of physics can be justified than for the occasions and uses of life; in a certain broad sense, indeed, there is no other justification. But the great majority of men must needs be practical in the narrow sense, and physics, as they study it, relates chiefly to the conditions which have been elaborated through the devices of industry as exemplified in our mills and factories, in our machinery of transportation, in optical and musical instruments, in the means for the supply of power, heat, light, and water for general and domestic use, and so on.

From this narrow practical point of view it may seem that there can be nothing very exacting in the study of the physical sciences; but was it physics? That is the question. One definition at least is to be repudiated; it is not "The science of masses, molecules and the ether." Bodies have mass and railways have length, and to speak of physics as the *science of masses* is as silly as to define railroading as the *practice of lengths*, and nothing as reasonable as this can be said in favor of the conception of physics as the science of molecules and the ether; it is the sickliest possible notion of physics, whereas the healthiest notion, even if a student does not wholly grasp it, is that physics is the science of the ways of taking hold of things and pushing them!

Bacon long ago listed in his quaint way the things which seemed to him most needful for the advancement of learning. Among other things he mentioned "*A New Engine or a Help to the mind corresponding to Tools for the hand*," and the most remarkable aspect of physical science is that aspect in which it constitutes a realization of this New Engine of Bacon. We continually force upon the extremely meager data obtained directly through our senses, an interpretation which, in its complexity and penetration, would seem to be entirely incommensurate with the data themselves, and we exercise over physical things a kind of rational control which greatly transcends the native cunning of the hand. The possibility of this forced interpretation and of this rational

control depends upon the use of two complexes: (*a*) A **logical structure**, that is to say, a body of mathematical and conceptual theory which is brought to bear upon the immediate materials of sense, and (*b*) a **mechanical structure**, that is to say, either (1) a carefully planned *arrangement of apparatus*, such as is always necessary in making physical measurements, or (2) a carefully planned *order of operations*, such as the successive operations of solution, reaction, precipitation, filtration, and weighing in chemistry.

These two complexes do indeed constitute a New Engine which helps the mind as tools help the hand, it is through the enrichment of the materials of sense by the operation of this New Engine that the elaborate interpretations of the physical sciences are made possible, and the study of elementary physics is intended to lead to the realization of this New Engine: (*a*) By the building up in the mind, of the logical structure of the physical sciences; (*b*) by training in the making of measurements and in the performance of ordered operations, and (*c*) by exercises in the application of these things to the actual phenomena of physics and chemistry at every step and all of the time with every possible variation.

That, surely, is a sufficiently exacting program; and the only alternative is to place the student under the instruction of Jules Verne where he need not trouble himself about foundations but may follow his teacher pleasantly on a care-free trip to the moon or with easy improvidence embark on a voyage of twenty-thousand leagues under the sea.

PART I.

MECHANICS.

"Our method is continually to dwell among things soberly, without abstracting or setting the mind farther from them than makes their images meet," and "the capital precept for the whole undertaking is that the eye of the mind be never taken off from things themselves, but receive their images as they truly are, and God forbid that we should offer the dreams of fancy for a model of the world."—BACON.

CHAPTER I.

WEIGHTS AND MEASURES.

1. Length. The *meter* is the length, at the temperature of melting ice, of a certain platinum bar which is preserved in the vaults of the International Bureau of Weights and Measures near Paris.* A very accurate copy † of this bar is deposited in the United States Bureau of Standards in Washington and this copy is the legal meter in the United States.

The *yard* is defined legally as 3600/3937 of a meter.‡

English Units of Length	*Metric Units of Length*
1 yard = 3 feet	1 kilometer = 1000 meters
1 foot = 12 inches	1 meter = 100 centimeters
1 mile = 1760 yards or 5280 feet	1 meter = 1000 millimeters

1 inch = 2.54 centimeters
1 centimeter = 0.3937 inch.

2. Angle. The angle all the way round a point, that is the angle which is represented by the entire circumference of a circle, is called a *perigon*. The perigon is a natural unit of angle and it is not necessary to preserve a material copy of it. The unit of angle which is universally used on divided circles is the *degree;* it is equal to 1/360 of a perigon.

*It was intended originally that the meter should be equal to one-ten-millionth part of the distance from the equator to either pole of the earth, but the extreme difficulty of reproducing accurate copies of the meter on the basis of this definition makes the definition not only impracticable but illusory.

† In fact there are three so-called *prototype meter-bars* in Washington. For a description of the International prototypes of the meter and of the kilogram, see *Nature*, Vol. 51, page 420, February 28, 1895.

‡On April 5, 1893, a decision was reached by the United States Superintendent of Weights and Measures, with the approval of the Secretary of the Treasury, that the meter and the kilogram would be regarded as the fundamental standards not only for metric units but also for the customary units of length and mass. See a History of the Standard Weights and Measures of the United States by Louis A. Fischer, Vol. I., pp. 365–381, *Bulletin of the Bureau of Standards* (United States Department of Commerce and Labor).

In many calculations it is convenient to express angle as follows: Imagine a circle of radius r drawn with its center at the apex of an angle, and let a be the length of the arc of the circle which is included between the boundaries of the angle; then the ratio a/r has a fixed value for a given angle, and the value of this ratio is frequently used as a numerical measure of the angle. The unit angle in this system is the angle of which the length of the subtending arc is equal to the radius and it is called the *radian.*

One perigon $= 2\pi$ radians $= 360$ degrees.

3. Area. The unit of area most extensively used is the area of a square of which the side is of unit length.*

English Units of Area

Areas are usually expressed in square inches, or square feet, or square yards.

Metric Units of Area

Areas are usually expressed in square centimeters or in square meters.

1 square inch	= 6.45 square centimeters.
1 square centimeter	= 0.155 square inches.

4. Volume or Capacity. The unit of volume most extensively used is the volume of a cube of which the edge is of unit length.

English Units of Volume

Volumes are frequently expressed in cubic inches, or in cubic feet or in cubic yards.

1 gallon = 4 quarts = 231 cubic inches

Metric Units of Volume

Volumes are frequently expressed in cubic centimeters, or in cubic meters.

1 liter = 1000 cubic centimeters

1 quart	= 1.36 liter
1 liter	= 0.88 quarts
1 cubic inch	= 16.38 cubic centimeters
1 cubic centimeter	= 0.061 cubic inch

5. Mass. Every one is familiar with the measurement of materials *by volume* and *by weight*, but every one does not distinguish between the two methods in common use for measuring "by weight," namely, (*a*) the method in which the *spring scale* is used, and (*b*) the method in which the *balance scale* is used. The spring scale measures the force with which the earth pulls a body; but the force with which the earth pulls a given body is slightly

*The *circular mil*, much used by electricians as a unit area, is the area of a circle one *mil* (1/1000 inch) in diameter. The area of any circle in circular mils is equal to the square of its diameter in mils.

different at different places on the earth, and therefore the weight of a given body, as indicated by a spring scale, varies with location. The indication of the balance scale, on the other hand, does not vary with the gravity-pull of the earth because the gravity-pull on the weights and the gravity-pull on the weighed body both change together. The indication of a balance scale is, therefore, independent of the value of gravity. *The result of the operation of weighing a body by a balance scale is called the mass of the body.**

Considering that weighing is nearly always done by the balance scale, it is evident that what is popularly called the weight of a body is what scientific men call the *mass* of the body, and it is important to remember that the force with which the earth pulls a body is called the *weight* of the body by scientific men. The verb *to weigh* nearly always means to determine the mass of a body by means of a balance scale.

Units of Mass. The kilogram is the mass of a certain piece of platinum which is deposited in Paris. A very accurate copy† of this kilogram is deposited in the United States Bureau of Standards‡ in Washington and this copy is the legal kilogram in the United States.

The *pound* avoirdupois is defined legally as 1/2.204622 of a kilogram.

English Units of Mass.	*Metric Units of Mass.*
1 ton = 2000 pounds	1 metric-ton = 1000 kilograms
1 pound = 16 ounces	1 kilogram = 1000 grams
1 pound = 7000 grains	1 gram = 1000 milligrams

*The mass of a body is sometimes defined on the basis of Newton's second law of motion. Thus a body may be said to have a mass of 90 pounds when a given unbalanced force acting on the body produces 1/90 as much acceleration as the same force would produce when acting as an unbalanced force upon a one-pound body. In all practical work, however, and in nearly all scientific work the mass of a body is determined by a balance scale and therefore the above definition of mass is the fundamental one because it refers to the method which is almost universally used for measuring mass.

†In fact there are three so-called *prototype kilograms* in Washington.

‡The Bureau of Standards is organized under the United States Department of Commerce and Labor, and the function of the Bureau is to verify weights and measures for commerical and scientific use and to carry out researches in various branches of physics and chemistry.

1 kilogram = 2.205 pounds
1 pound = 0.4536 kilogram.
$62\frac{1}{2}$ pounds of water = 1 cubic foot nearly.
1 gram of water = 1 cubic centimeter very nearly.

Measurement of mass. The analytical balance consists of a delicately mounted equal-arm lever with pans suspended from its ends. *The balance is used simply for indicating the equality of the masses of two bodies*, that is, two bodies are said to have equal masses when they balance each other when suspended from the ends of an equal-arm lever.

The determination of the mass of a body by means of the balance depends upon the use of a *set of weights* which may be combined in such a way as to match the mass of the body. Such a set of weights may be made by taking two pieces of metal weighing together one kilogram and then making them balance each other by cutting metal off from one and adding the shavings to the other, thus giving a half-kilogram weight. Then a quarter-kilogram weight may be made in the same way and so on. A set of weights more convenient in use is a set which contains a five, a two, and two ones of each — units, tens, hundreds, etc., of grams.

6. Force. The effort exerted to start a body moving or the effort which must be continually exerted to keep a body in motion when it is opposed by friction is called *force*.

One *pound of force* is the force with which the earth pulls on a one-pound body. Thus, to say that a powerful locomotive exerts a force of 20,000 pounds in starting a train means that the pull of the locomotive is 20,000 times as great as the pull of the earth on a one-pound body.

One *kilogram of force* is the force with which the earth pulls on a one-kilogram body.

It is evident that the word pound (or kilogram) has a very different meaning in the following two cases, (*a*) when we speak of a pound of sugar or 1000 pounds of coal, and (*b*) when we speak of a force of one pound or a force of 1000 pounds. One *pound of sugar* is a perfectly definite amount of sugar, but the pull of

the earth on a one-pound body is not the same everywhere so that the *pound of force* is not a perfectly definite* amount of force.

7. Density and specific gravity. Every one knows that lead is heavier than cork and yet everyone feels instantly that the question "Which is heavier, a pound of lead or a pound of cork?" is ambiguous. The word heavy has, in fact, two meanings. One pound of cork is heavier than one-half pound of lead in the same sense that a ton of coal is heavier than a pound of coal. In this case the word heavy refers to the *amount of material* expressed in pounds or kilograms. On the other hand lead is heavier than cork in the sense that a piece of lead weighs more than an equal bulk of cork. In this case the word heavy refers to an *inherent property of a substance*; the word *density* is used to designate this inherent property of heaviness. Thus lead has a greater density than cork. The density of a substance may be specified by giving its mass per unit volume. Thus the density of water is about 62½ pounds per cubic foot, the density of copper is 555 pounds per cubic foot, and the density of ordinary kerosene is about 7 pounds per gallon. In scientific work it is usual to specify the density of a substance in grams per cubic centimeter.

The *specific gravity* of a substance at a given temperature is the ratio of the density of the substance to the density of water at the same temperature. Thus if a substance is 2.5 times as heavy as an equal bulk of water, the specific gravity of the substance is said to be 2.5.

8. Time. The length of a day (noon to noon) is slightly different at different seasons of the year, but the average length of a day from year to year is very nearly invariable. The *second* is 1/86400th of the average day (noon to noon).†

*The *dyne* and the *poundal* are perfectly definite units of force. This matter is discussed at some length in Art. 33. It has been suggested that the weight of a pound at a specified place on the earth be adopted as the definite unit of force, and that a force in pounds measured at any other place on the earth be multiplied by the ratio g'/g where g is the acceleration of gravity at the standard place and g' is the acceleration of gravity at the given place.

†For a full discussion of the measurement of time see Young's *General Astronomy*, page 15, Ginn & Co.

Measurement of time. Any movement of a body which repeats itself in equal intervals of time is called *periodic motion*; single movements are called *vibrations*. A vibrating pendulum is the most familiar example of periodic motion. If one counts the number N of vibrations of a pendulum in a day and the number n during a given interval of time, then the interval is equal to n/N of a day.

The clock* is a machine for *maintaining* the vibrations of a pendulum and for *keeping count* of the vibrations. In the simplest case the pendulum is adjusted to make 86,400 vibrations in a day (one vibration each second), and the clock hands count these vibrations in groups of *sixty* and *sixty-times-sixty* instead of counting in tens, hundreds and thousands.

In portable clocks a *balance wheel* takes the place of a pendulum.

PROBLEMS.

1. Reduce an angle of 20° to radians. Reduce an angle of one radian to degrees. Ans. 20° equals 0.349 radian; one radian equals 57°.2958.

2. (*a*) Find the area in circular mills of a wire of which the diameter is 0.1 inch. (*b*) Find the area in circular mils of a copper bar ⅛ of an inch thick and ½ inch wide. Ans. (*a*) 10,000 circular mils; (*b*) 79580 circular mils.

Note. The area of a rectangle in square mils is found by multiplying together the length and breadth of the rectangle in mils, and the result may be reduced to circular mils by multiplying by $4/\pi$.

3. The density of alcohol is 6.35 pounds per gallon. What is its density in pounds per cubic inch? What is its density in grams per cubic centimeter? Ans. 0.0274 pound per cubic inch; 0.76 gram per cubic centimeter.

4. The specific gravity of iron is 7.8. What is its density in pounds per cubic inch? What is the mass, in pounds, of an iron

*For discussion of clock errors and description of clock escapements see Encyclopedia Britannica, 9th Edition, article *Clock*. For description of compensated chronometer balance wheel and chronometer escapement see Lockyer's book entitled *Stargazing*, pages 175 to 210.

bar 30 feet long and $5\frac{1}{2}$ square inches sectional area? Ans. 0.282 pound per cubic inch; the bar has a mass of 558 pounds.

Note. The foundryman's rule for finding the mass in pounds of an iron casting is to divide the volume of the casting in cubic inches by 4. This is equivalent to taking the density of cast iron as $\frac{1}{4}$ of a pound per cubic inch.

5. A bottle weighs 50.62 grams empty and 288.93 grams when full of water at 21° C. What is the cubic contents of the bottle? Neglect the buoyant force of the air. Ans. 238.77 cubic centimeters.

Note. The density of water at 21° C. is 0.99806 gram per cubic centimeter.

6. The bottle which is specified in problem 5 weighs 239.2 grams when filled with oil at 21° C. What is the density of the oil? Neglect the buoyant force of the air. Ans. 0.79 gram per cubic centimeter.

CHAPTER II.

PHYSICAL ARITHMETIC.

9. Measures; units. In the expression of a physical quantity two factors always occur, a numerical factor and a unit. The numerical factor is called the *measure* of the quantity. Thus a certain length is 65 centimeters, a certain time interval is 250 seconds, a certain electric current is 25 amperes, a certain electromotive force is 110 volts.

It is a great help towards a clear understanding of physical calculations to consider that both *units* and *measures* are involved in a product of two physical quantities or in a quotient of two physical quantities. Thus a rectangle is 5 centimeters wide and 10 centimeters long; and its area is 5 centimeters times 10 centimeters, which is equal to 50 *square centimeters*. A cylinder is 10 centimeters long and the area of one of its ends is 25 square centimeters; and its volume is 10 centimeters times 25 square centimeters, which is equal to 250 *cubic centimeters*. A train travels 500 feet in 10 seconds and its average velocity during the time is 500 feet divided by 10 seconds which is equal to 50 *feet per second*. A body is dragged through a distance of 15 feet by a force of 10 pounds and the amount of work done is 15 feet times 10 pounds, which is equal to 150 *foot-pounds*. The word *per* connecting the names of two units indicates that the unit following is a *divisor*, thus a velocity of 50 feet *per* second may be, and often is written 50 feet/second. A hyphen connecting the names of two units indicates a *product* of the units; products and quotients of units arrived at in this way are always *new* physical units. Thus the foot per second is a unit of velocity, the foot-pound is a unit of work.

It is important to carry the units through with every numerical calculation, the arithmetical operations among the various units

being indicated algebraically. When this is done there can be no ambiguity as to the meaning of the result, and when this is not done the result has, strictly speaking, no physical meaning at all.

Although the unit in terms of which a result is expressed is known* when the units are carried through a numerical calculation, it frequently happens that the unit is so entirely novel that it might almost as well be unknown. Thus the rule for finding the area of a rectangle by taking the product of length and breadth is entirely general, no matter what units of length are used, and the area of a rectangle 2 meters long and 50 centimeters wide is equal to 2 meters × 50 centimeters or 100 meter-centimeters. Now the *meter-centimeter* is a unit of area equal to the area of a rectangle one meter long and one centimeter wide and it is so entirely unfamiliar as a unit of area among men engaged in practical work that one might almost as well not know the value of an area at all as to have it given in terms of such a unit. **It is, for this reason, nearly always necessary to reduce the data of a problem to certain accepted units before these data can be used intelligibly in numerical calculations.**

10. Units, fundamental and derived. The *fundamental physical units* are those which are fixed by arbitrary preserved standards. Thus the unit of length is preserved as a platinum bar in Paris, the unit of mass is preserved by a piece of platinum in Paris, and the second is naturally preserved in the constancy of speed of rotation of the earth.†

Derived physical units are those which are defined in terms of the fundamental units and of which no material standard need be preserved. Thus the unit of area is defined as the area of a square of which each side is a unit of length, and there is no need of preserving a material standard of the unit of area. The unit of velocity is defined as unit distance traveled per second,

*See Art. 12, on dimensions of derived units.

†The standard candle is a fundamental unit because it is fixed by an arbitrary preserved standard, and a degree centigrade is a fundamental unit because it is preserved in the constant temperature-difference between freezing water and boiling water.

and there is no need of preserving a material standard of the unit of velocity, indeed, it would be impracticable to preserve a velocity.

Remark 1. Quantities such as area, volume, velocity, electric current, etc., for which derived units are used, may be called *derived quantities* for the reason that they are defined (as quantities) in terms of the *fundamental quantities*, length, mass, and time. For example, the density of a body is defined as the ratio of its mass to its volume; the velocity of a body is defined as the quotient obtained by dividing the distance traveled during an interval of time, by the interval, etc.

Remark 2. The choice of fundamental units is a matter which is governed solely by practical considerations; in the first place the fundamental units must be easily preserved as material standards, and in the second place the fundamental quantities must be susceptible of very accurate measurement, for the definition of a derived unit cannot be *realized** with greater accuracy than the fundamental quantities can be measured.

11. The c. g. s. system of units.† Derived units based upon the *centimeter* as the unit length, the *gram* as the unit mass, and the *second* as the unit time, are in common use. This system of derived units is called the c. g. s. (centimeter-gram-second) system.

Thus the square centimeter is the c. g. s. unit of area, the cubic centimeter is the c. g. s. unit of volume, one gram per cubic centimeter is the c. g. s. unit of density, one centimeter per second is the c. g. s. unit of velocity, etc.

*The definition of a physical quantity is always an actual physical operation. Thus the mass of a body is defined by the operation of weighing with a balance; the density of a body is defined by the operation involved in the finding of mass and volume, for mass and volume must be determined before mass can be divided by volume to give density.

†So long as the English units of length and mass continue to be used, it will be necessary for engineers to use the units of the f. p. s. (foot-pound-second) system to some extent, although these systematic f. p. s. units are, many of them, never used in commercial work. Thus the f. p. s. unit of force is the *poundal*, the f. p. s. unit of work is the *foot-poundal*.

Practical units. In many cases the c. g. s. unit of a quantity is either inconveniently small or inconveniently large so that the use of the c. g. s. unit would involve the use of very awkward numbers. Thus the power required to drive a small ventilating fan is 500,000,000 ergs per second, the electrical resistance of an ordinary incandescent lamp is 220,000,000,000* c. g. s. units of resistance, the capacity of an ordinary Leyden jar is 0.000,000,000,000,000,005 c. g. s. units of capacity. In such cases it is convenient to use a multiple or a sub-multiple of the c. g. s. unit as a *practical* unit. Thus 5×10^8 ergs per second is equal to 50 watts, 22×10^{10} c. g. s. units of resistance is 220 ohms, 5×10^{-18} c. g. s. units of electrostatic capacity is equal to 0.005 micro-farad.

Legal units. The system of units now in general use presents several cases in which the fundamental measurement of a derived quantity in terms of length, mass, and time is extremely laborious and not very accurate at best. Thus the measurement of electrical resistance in terms of length, mass, and time is very difficult, whereas the measurement of electrical resistance in terms of the resistance of a given piece of wire is very easy indeed and it may be carried out with great accuracy. In every such case the fundamental measurement is carried out once for all with great care and the best possible material copy is made of the derived unit and this copy is adopted as the standard legal unit.

12. Dimensions of derived units. The definition of a derived unit always implies an equation which involves the derived unit together with one or more of the fundamental units of length, mass, and time. This equation solved for the derived unit is said to express the dimensions of that unit.† Thus the velocity of a body is defined as the quotient l/t, where l is the distance traveled by the body during the interval of time t, so that the unit of velocity is equal to the unit of length divided by the unit of time.

Examples. Let l be the unit of length, m the unit of mass, and t the unit of time. Then the unit of area is equal to l^2, the

*In the writing of very large or very small numbers it is always more convenient and more intelligible to use a positive or negative power of 10 as a factor. Thus 220,000,000,000 is best written as 22×10^{10}, and 0.000,000,000,000,000,005 is best written as 5×10^{-18}.

†And also the dimensions of the derived quantity.

unit of volume is equal to l^3, the unit of density is equal to m/l^3 the unit of velocity is equal to l/t, the unit of force is equal to ml/t^2, the unit magnetic pole is equal to $\sqrt{ml^3}/t$, etc.

Naming of derived units. Many derived units have received specific names. Such are the *dyne*, the *erg*, the *ohm*, the *ampere*, the *volt*, etc. Those derived units which have not received specific names are specified by writing, or speaking-out, their dimensions. Thus the unit of area is the *square centimeter*, the unit of density is the *gram per cubic centimeter*, the unit of velocity is the *centimeter per second*, the unit of momentum is the *gram-centimeter per second* (written gr. cm./sec.). In the case of units which have complicated dimensions this method is not convenient in speech. Thus we specify a certain magnetic pole as 150 gr.$^{\frac{1}{2}}$ cm.$^{\frac{3}{2}}$/sec. (spoken, 150 *c. g. s. units pole*). Some units have dimensions so simple that to specify them in terms of their dimensions is almost meaningless. Thus an angle has zero dimensions, being a length divided by a length (an arc divided by a radius). An angular velocity is an angle divided by time, and dimensionally an angular velocity is equal to $1/t$. In such cases it is impossible to name a unit in terms of its dimensions.

13. Scalar and vector quantities. A *scalar quantity* is a quantity which has magnitude only. Thus everyone recognizes at once that to specify 10 cubic meters of sand, 25 kilograms of sugar, 5 hours of time, is, in each case, to make a complete specification. Volume, mass, time, energy, electric charge, etc., are scalar quantities.

A *vector quantity* is a quantity which has both magnitude and direction, and to specify a vector one must give both its magnitude and direction. This necessity of specifying both the magnitude and direction of a vector is especially evident when one is concerned with the relationship of two or more vectors. Thus if one travels a stretch of 10 kilometers and then a stretch of 5 kilometers more, he is by no means necessarily 15 kilometers from home; his position is, in fact, indeterminate until the direction of each stretch is specified. If one man pulls on a car with a force

A of 200 units and another pulls with a force B of 100 units, the total force acting on the car is by no means necessarily equal to 300 units. In fact, the total force is unknown both in magnitude and direction until the direction as well as the magnitude of each force A and B is specified. Length, velocity, acceleration, momentum, force, magnetic field intensity, etc., are vector quantities.

Representation of a vector by a line. In all discussions of physical phenomena which involve the relationships of vectors, it is a great help to the understanding to represent the vectors by lines. Thus in the discussion of the combined action of several forces on a body, it is a great advantage to represent each force by a line. To represent a vector by a line, draw the line in the direction of the vector (from any convenient point) and make the length of the line proportional to the magnitude of the vector. Thus if a northward velocity of 600 centimeters per second of a moving body is to be represented by a line, draw the line towards the north and let each unit length of line represent a chosen number of units of velocity.

When a vector α is represented by a line, the line is parallel to α and the value of α is given by the equation

$$\alpha = S \cdot l$$

in which l is the length of the line and S is the number of units of α represented by each unit length of the line. The quantity S is called the *scale* to which the line represents the vector α.

14. Addition of vectors. The addition polygon. Many cases arise in physics where it is necessary to consider the *single force* which is equivalent to the combined action of several given forces; where it is necessary to consider the *single actual velocity* which is equivalent to several given velocities each produced, it may be, by a separate cause; where it is necessary to consider the *single actual intensity* of a magnetic field due to the combined action of several causes each of which alone would produce a magnetic field of given direction and intensity; and so on. *The*

single vector is in each case called the vector-sum, or resultant, of the several given vectors. Scalar quantities are added by the ordinary methods of arithmetic, thus 10 pounds of sugar plus 15 pounds of sugar is 25 pounds of sugar; but the addition of several vectors is not an arithmetical operation, it is a geometrical operation, and it is for this reason that the addition of vectors is sometimes called geometric addition.

The simplest vector quantity is the movement of a body over a given distance in a given direction. Such a movement is called a *displacement* of the body. Thus the arrow n in Fig. 1 represents a given northward displacement and the arrow e represents a given eastward displacement. If a body moves northward as represented by the arrow n and then eastward as represented by the arrow e (or if it moves eastward as represented by the arrow e and then northward as represented by the arrow n) it will reach the point P. Therefore the displacement OP is spoken of as the vector sum or resultant of the two displacements n and e.*

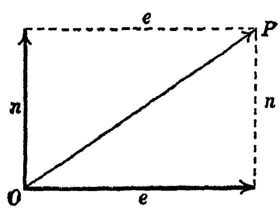


Fig. 1.

Addition of two forces. The parallelogram of forces. Let the lines a and b, Fig. 2, represent two forces acting upon any body, a boat for example. The vector-sum, or resultant, of the two forces a and b is represented by the diagonal r of the parallelogram of which a and b are the sides. It is evident that the

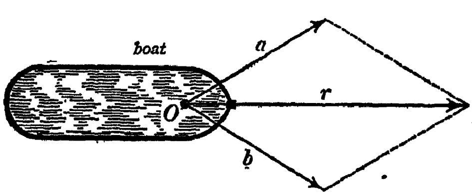


Fig. 2.

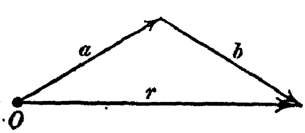


Fig. 3.

*The arrows n and e in Fig. 1 are drawn at right angles to each other for the sake of simplicity. The above statement is true, however, whatever the angle between n and e may be.

geometric relation between a, b and r is completely represented by the triangle in Fig. 3, in which the line which represents the force b is drawn from the extremity of the line which represents the force a.

*Experimental verification of the parallelogram of forces.** The experimental verification of the parallelogram of forces is usually given as a laboratory exercise in elementary physics. Figure 4

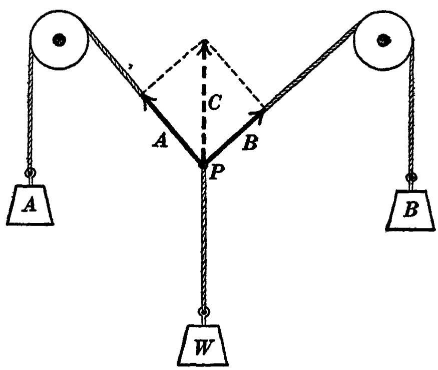


Fig. 4.

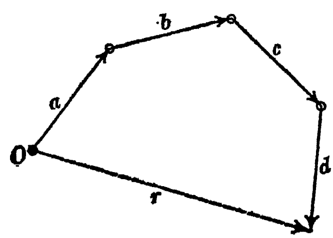


Fig. 5.

shows an arrangement which is sometimes used for this purpose. Two known forces A and B are made to act at the point P, and their resultant C is known to be upwards and equal to the weight W.

Addition of any number of forces. The force polygon. Given a number of forces a, b, c and d. Draw the line which represents the force a from a chosen point O, Fig. 5, draw the line which represents the force b from the extremity of a, draw the line which represents c from the extremity of b, and draw the line which represents the force d from the extremity of c. Then the line from O to the extremity of d represents the geometric sum of the forces a, b, c and d.

*It is possible to reduce the addition of two simple vectors like two forces or two velocities to the addition of two displacements. The argument involved in this reduction constitutes a mathematical proof of the parallelogram of forces. One form of this mathematical proof is given in Art. 60.

To prove the above statement concerning the force polygon consider a number of forces which are to be added, add forces No. 1 and 2 by using the parallelogram of forces, add No. 3 to the resultant of No. 1 and 2 in the same way, add No. 4 to the resultant of No. 1, 2 and 3 in the same way, and so on, and the truth of the above statement will appear at once in the diagram so constructed.

The vector sum of a number of forces is equal to zero if the forces are parallel and proportional to the sides of a closed polygon, the directions of the forces being in the directions in which the sides of the polygon would be traced in going round the polygon.*

A particular case of this general proposition is that the vector sum or resultant of three forces is equal to zero if the three forces are parallel and proportional to the three sides of a triangle and in the directions in which the sides would be passed over in going round the triangle.

Note. What is said above concerning the addition of forces applies to the addition of vectors of any kind, velocities, accelerations, magnetic field intensities and so on.

15. Resolution of vectors. Any vector may be replaced by a number of vectors of which it is the vector sum, or in other words a given vector can be resolved into parts. The simplest case is that in which a vector is replaced by two vectors; the two vectors must be parallel and proportional to the respective sides of a parallelogram of which the diagonal represents the given vector. The following discussion of the resolution of forces illustrates this matter.

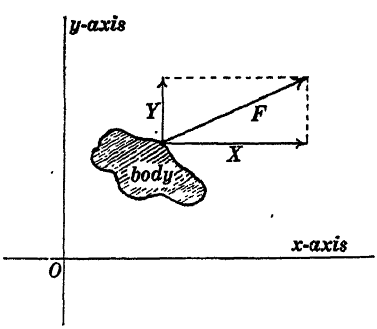


Fig. 6.

Consider force F in Fig. 6. The two forces X and Y would have exactly the same effect as the force F and therefore they

*The sides of a force polygon need not all lie in a plane except, of course, when the polygon is a triangle.

may be thought of in place of the force F. The two forces X and Y in Fig. 6 are called the *rectangular components* of the force F because the angle between X and Y is a right angle. Or, in other words, if a rectangle be constructed whose diagonal represents a given force, then the sides of the rectangle represent what are called the rectangular components of that force in the directions of those sides.

Example. Consider the force F, Fig. 7, which is pulling on a car, the car being constrained to move along the track. The force F is equivalent to the two forces α and β combined. The

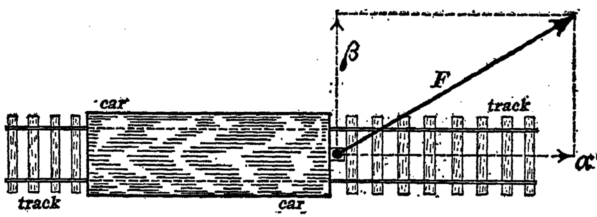


Fig. 7.

force β has no effect in moving the car (it may have an indirect effect on the motion of the car by pulling the flanges of the wheels against the rails and thus increasing the friction) so that the force α is the part of F which is effective in moving the car. This force α is called the *resolved part* of F, or the *component* of F in the direction of the track.

16. Scalar and vector products and quotients.—The product, or quotient, of a scalar and a vector, α, is another vector parallel to α.

Examples.—The distance l traveled by a body in time t is equal to the product vt, where v is the velocity of the body; and l is parallel to v.

The force F with which a fluid pushes on an exposed area a is equal to the product pa where p is the hydrostatic pressure of the fluid (a scalar). The vector direction of an area is the direction of its normal and this is parallel to F.

A force F which does an amount of work W in moving a body a distance d (which is parallel to F) is equal to W/d.

The acceleration of a body is equal to $\Delta v/\Delta t$ where Δv is the increment of velocity in the time interval Δt; the acceleration is a vector which is parallel to Δv.

Vector products. Case I. *Parallel vectors.* The product or quotient of parallel vectors is a scalar. Thus $W=Fd$, in which W is the work done by a force F acting through a distance d in its direction; $p=F/a$, in which p is the pressure

in a liquid which exerts a force F on an exposed area a; $V=la$, in which V is the volume of a prism of base a and altitude l.

CASE II. *Orthogonal vectors.* The product, or quotient, of two mutually perpendicular vectors is a third vector at right angles to both factors. Thus $a=lb$ and $b=a/l$, in which a is the area of a rectangle of length l and breadth b; $T=Fl$, in which T is the *moment* or *torque* of a force F, and l is its arm. The product of a vector and a line perpendicular thereto is called the *moment* of the vector.

CASE III. *Oblique vectors.* The product of two oblique vectors consists of two parts, one of which is a scalar and the other is a vector. Consider two vectors, α and β, Fig. 8. Resolve β into two components, β' and β'' respectively parallel to and perpendicular to α. Then

$$\alpha\beta=\alpha(\beta'+\beta'')=\alpha\beta'+\alpha\beta'',$$

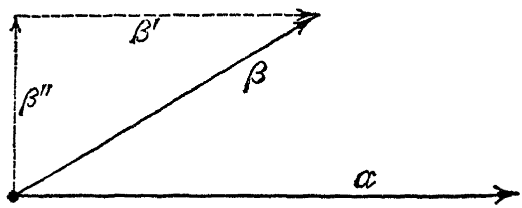


Fig. 8.

in which $\alpha\beta'$ is a scalar and $\alpha\beta''$ is a vector. The scalar part of a vector product is indicated thus, $S\cdot\alpha\beta$ (read *scalar-alpha-beta*). The vector part of a vector product is indicated thus, $V\cdot\alpha\beta$ (read *vector-alpha-beta*). When $V\cdot\alpha\beta=0$, α and β are parallel; when $S\cdot\alpha\beta=0$, α and β are orthogonal.

Examples of products of vectors. The area of any parallelogram is equal to $V\cdot bl$ where b and l are the two sides of the parallelogram. The work done by a force is equal to $S\cdot Fd$ where F is the force and d is the displacement of the point of application of the force. The volume of any parallelopiped is equal to $S\cdot al$ where a is the area of the base and l is the length of the other edge. The torque action of any force is equal to $V\cdot Fr$ where F is the force and r is the lever arm.

17. Constant and variable quantities. The study of those physical phenomena which are associated with unvarying conditions is comparatively simple, whereas the study of those phenomena which are associated with rapidly varying conditions is generally very complicated. Thus, to design a bridge is to so proportion its members that a minimum amount of material may be required to build it, and it is a comparatively simple problem to design a bridge to carry a steady load because it is easy to calculate the stress in each member due to a steady load, but it is an extremely complicated problem to design a bridge to carry

a varying load, such as a moving locomotive which comes upon the bridge suddenly and moves rapidly from point to point.

Two kinds of variations are to be distinguished, namely, variations in space and variations in time.

Variations in time. In the study of phenomena which depend upon conditions which vary in time, that is, upon conditions which vary from instant to instant, it is necessary to direct the attention to what is taking place at this or that instant; or, in other words, to direct the attention to what takes place during very short intervals of time; or, borrowing a phrase from the photographer, to make snap-shots, as it were, of the varying conditions.

Definition of rate of change. Principle of continuity.* In order to establish the rather difficult idea of instantaneous rate of change of a varying quantity, it is a great help to make use of a simple physical example. Therefore let us consider a pail out of which water is flowing through a hole in the bottom. Let x be the amount of water in the pail. Evidently x is a changing quantity. Let Δx be the amount of water which flows out of the pail during a given interval of time Δt, then the quotient $\Delta x/\Delta t$ is called *the average rate of change of x* during the given interval of time, and, if the interval Δt is very short, the quotient $\Delta x/\Delta t$ approximates to what is called *the rate of change of x at a given instant, or the instantaneous rate of change of x.*

*Nearly everyone falls into the idea that such an expression as 10 feet per second means 10 feet of actual movement in an actual second of time, but a body moving at a velocity of 10 feet per second, might not continue to move for a whole second, or its velocity might change before a whole second has elapsed. Thus, a velocity of 10 feet per second is the same thing as a velocity of 864,000 feet per day, but a body need not move steadily for a whole day in order to move at a velocity of 864,000 feet per day. Neither does a man need to work for a whole month to earn money at the *rate* of 60 dollars per month, nor for a whole day to earn money at the *rate* of 2 dollars per day. A falling body has a velocity of 19,130,000 miles per century after it has been falling for one second, but to specify its velocity in miles per century does not mean that it moves as far as a mile or that it continues to move for a century! The units of length and time which appear in the specification of velocity are completely swallowed up, as it were, in the idea of velocity, and the same thing is true of the specification of any rate.

The amount of water which flows out of the pail during a short interval of time is very nearly proportional to the time; and **if a shorter and shorter time interval is considered the amount of water which flows out of the pail is more and more nearly in exact proportion to the time** so that the quotient $\Delta x/\Delta t$ approaches a perfectly definite finite value as Δt and Δx both approach zero. If the amount of water in the pail were to change by sudden jumps as it were, then the amount of water flowing out of the pail in a very short interval would not be even approximately proportional to the interval, and the rate of change of x at a given instant would be unthinkable; but physical quantities which vary in value from instant to instant always vary continuously and have, therefore, at each instant a definite rate of change; that is to say, the quotient $\Delta x/\Delta t$ always approaches a definite limiting value as the interval Δt is made shorter and shorter.

Let it be understood that this paying attention to what takes place during very short intervals of time does not refer to *observation* but to *thinking*, it is a matter of mathematics,* and therefore the following purely mathematical illustration is a legitimate example.

Example. Consider a square of which the sides are growing in proportion to elapsed time so that the length of each side of

*Two distinct methods are involved in the directing of the attention to what takes place during infinitesimal time intervals or infinitesimal regions of space.

(*a*) *The method of differential calculus.* A phenomenon may be prescribed as a pure assumption and the successive instantaneous aspects derived from this prescription. Thus we may prescribe uniform motion of a particle in a circular path, and then proceed to analyze this prescribed motion as exemplified in Art. 38; or we may prescribe a uniform twist in a cylindrical metal rod, and then proceed to analyze the prescribed distortion (Art. 135). This method is also illustrated by the example given in the text, in which the expression for the growing sides of a square is prescribed.

(*b*) *The method of integral calculus.* It frequently happens that we know the action which takes place at a given instant or in a small region, and can formulate this action without difficulty, and then the problem is to build up an idea of the result of this action throughout a finite interval of time, or throughout a finite region of space. For example, a falling body gains velocity at a known constant rate at each instant, how much velocity does it gain and how far does it travel in a given finite interval of time?

the square may be expressed as kt, where k is a constant and t is elapsed time reckoned from the instant when one of the sides is equal to zero. Then the area of the square is $S = k^2t^2$, and it is evident that the area is increasing. Let $t + \Delta t$ be written for t in the expression for S and we have

$$S + \Delta S = k^2(t + \Delta t)^2 = k^2t^2 + 2k^2t \cdot \Delta t + k^2(\Delta t)^2$$

whence, subtracting $S = k^2t^2$, member from member, we have

$$\Delta S = 2k^2t \cdot \Delta t + k^2(\Delta t)^2$$

or

$$\frac{\Delta S}{\Delta t} = 2k^2t + k^2 \cdot \Delta t$$

from which it is evident that $\Delta S/\Delta t$ becomes more and more nearly equal to $2k^2t$ as Δt is made smaller and smaller. The value of $\Delta S/\Delta t$ for an indefinitely small value of Δt is usually represented by the symbol* dS/dt so that we have

$$\frac{dS}{dt} = 2k^2t$$

Propositions concerning rates of change. (*a*) Consider a quantity x which changes at a constant rate a, then the total change of x during time t is equal to at. For example, a man earns money at the rate of 2 dollars per day and in 10 days he earns 2 dollars per day multiplied by 10 days which is equal to 20 dollars. A falling body gains 32 feet per second of velocity every second, that is, at the constant rate of 32 feet per second per second, and in 3 seconds it gains 32 feet per second per second multiplied by 3 seconds which is equal to 96 feet per second.

(*b*) Consider a quantity y which always changes k times as fast as another quantity x, then, if the two quantities x and y start from zero together, y will be always k times as large as x. That

*The symbol dS/dt is one single algebraic symbol and it is not to be treated otherwise. It stands for the *rate of change of S at any given instant* and is to be so read.

is, if $dy/dt = k \cdot dx/dt$, then $y = kx$ if y and x start from zero together.

Conversely, if one quantity is always k times as large as another it must always change k times as fast.

(*c*) Consider a quantity s which is equal to the sum of a number of varying quantities x, y and z, then the rate of change of s is equal to the sum of the rates of change of x and y, and z. This may be shown as follows: let Δx, Δy and Δz be the increments of x, y and z during a given interval of time Δt, then the increment of s is

$$\Delta s = \Delta x + \Delta y + \Delta z$$

whence, dividing both members by Δt we have

$$\frac{\Delta s}{\Delta t} = \frac{\Delta x}{\Delta t} + \frac{\Delta y}{\Delta t} + \frac{\Delta z}{\Delta t}$$

or, if the interval Δt is very short, we have

$$\frac{ds}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} \tag{i}$$

Conversely, if the relation (i) is given, that is, if s is a quantity whose rate of change is known to be equal to the sum of the rates of change of x, and y, and z, then s must be equal to $x + y + z$ if s, x, y, and z all start from zero together.

Variations in space. Imagine a bar of iron one end of which is red hot and the other end of which is cold. Evidently the temperature of the bar varies from point to point. The pressure in a vessel of water increases more and more with the depth, the density of the air decreases more and more with increasing altitude above the sea.

Such quantities as temperature, pressure, and density which refer to the physical conditions at various points in a substance are called *distributed quantities*. The distribution is said to be *uniform* when the quantity has the same value throughout a substance, the distribution is said to be *non-uniform* when the quan-

tity varies in value from point to point. Thus the temperature of the air in a room is uniform if it is the same throughout, whereas the temperature of a bar of iron which is red hot at one end and cold at the other end is non-uniform.

In the study of phenomena dependent upon conditions which vary from point to point in space, *the attention must be directed to what takes place in very small regions*, because too much takes place in a finite region. Let it be understood, however, that this paying attention to what takes place in small regions does not refer to *observation* but to *thinking*, it is a matter of mathematics, and therefore the following illustration is a legitimate example, even though the physics may not be entirely clear:

A rigid wheel rotates at a speed of n revolutions per second. Let us consider what is called the kinetic energy of the wheel. Now the kinetic energy in ergs of a moving body is equal to $\frac{1}{2}mv^2$, where m is the mass of the body in grams and v is its velocity in centimeters per second; but the difficulty here is that the different parts of the wheel have different velocities, and if we are to apply the fundamental formula for kinetic energy ($= \frac{1}{2}mv^2$) to a rotating wheel it is necessary to consider each small portion of the wheel by itself. Thus, a small portion of the wheel at a distance r from the axis has a velocity which is equal to $2\pi nr$, and if we represent the mass of the small portion by Δm, the kinetic energy of the portion will be $\frac{1}{2} \times \Delta m \times (2\pi nr)^2$, or $2\pi^2 n^2 \cdot r^2 \Delta m$; so that the total kinetic energy of the wheel will be equal to the sum of a large number of such terms as this. But, the factor $2\pi^2 n^2$ is common to all the terms, and therefore the total kinetic energy is equal to $2\pi^2 n^2$ times the *sum of a large number of terms like* $r^2 \Delta m$. This sum is called the moment of inertia of the wheel.

Gradient. Consider an iron bar of which the temperature is not uniform. Let ΔT be the difference of the temperatures at two points distant Δx from each other, then the quotient $\Delta T / \Delta x$ is called the *average* temperature grade or gradient along the stretch Δx, and if Δx is very small the quotient $\Delta T / \Delta x$ is called the actual temperature gradient and it is represented by the

symbol dT/dx. The use of this idea of temperature gradient is illustrated in the discussion of the conduction of heat.

18. Varying vectors.* The foregoing article refers solely to varying scalar quantities. The mathematics of varying vectors may also be considered in two parts, namely time variation and space variation.

Time variation of velocity.† The velocity of a body is defined as the distance traveled in a given time divided by the time. When a velocity always takes place in a fixed direction it may be thought of as a purely scalar quantity. Thus a falling body has a velocity of 50 feet per second at a given instant and 3 seconds later it has a velocity of 146 feet per second, so that the increase of velocity in three seconds in 96 feet per second and the rate of increase is 96 feet per second divided by 3 seconds which is equal to 32 feet per second per second; but, suppose that the velocity of a body at a given instant is 50 units in a specified direction and that 3 seconds later it is 146 units in some other specified direction, then the change of velocity is by no means equal to 96 units and the rate of change of the velocity is by no means equal to 32 units per second.

The rate of change of the velocity of a body is called the *acceleration* of the body. Consider any moving body, a ball tossed through the air for example, let its velocity, v_1, at a given instant be represented by the line OA, Fig. 9*a*, let its velocity v_2 at a later instant be represented by the line OB, and let the elapsed time interval be Δt. Now the velocity which must be

*This branch of mathematics is very largely ignored in present undergraduate courses, and yet no one can have a clear insight into the phenomena of motion without having an idea of the time variations of velocity, and no one can have a clear insight into the phenomena of fluid motion and of electricity and magnetism without having some understanding of the space variation of such vectors as fluid velocity, magnetic field, and electric field.

†What is here stated concerning the time variation of velocity applies to the time variation of any vector whatever. Thus if any varying vector is represented to scale by a line drawn from a fixed point, then the velocity of the end of the line represents the rate of change of the vector to the same scale that the line itself represents the vector.

added (geometrically) to v_1 to give v_2 is the velocity Δv which is represented by the line AB. Therefore the, change of the velocity of the tossed ball during the interval Δt is the vertical

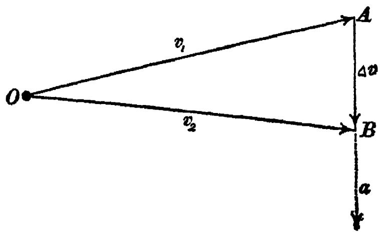


Fig. 9*a*.

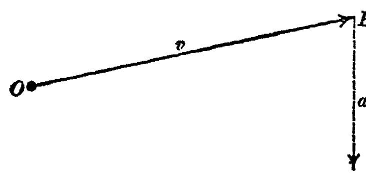


Fig. 9*b*.

velocity Δv shown in the figure, and the acceleration, a, of the ball is equal to $\Delta v/\Delta t$ which is of course in the direction of Δv. If the varying velocity of a tossed ball be represented by a line OP, Fig. 9*b*, drawn from a fixed point O; then, as the velocity changes the line OP will change, the point P will move, *and the velocity of the point P will represent the* **acceleration of the body** *to the same scale that the line OP represents the* **velocity of the body.**

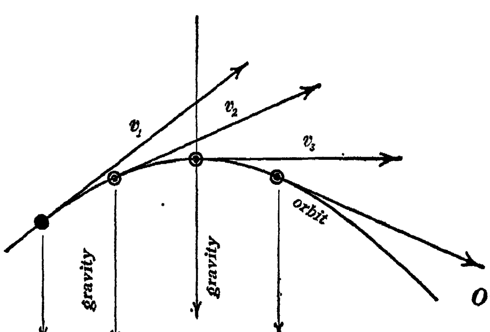


Fig. 10*a*.

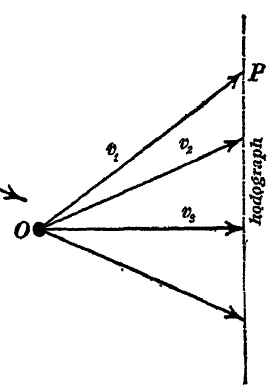


Fig. 10*b*.

The orbit of a moving body is the path which the body describes in its motion. Thus the orbit of a tossed ball is a parabola as shown in Fig. 10*a*, and the orbit of a ball which is twirled

on a cord is a circle as shown in Fig. 11*a*. Let the line OP, Fig. 10*b* or Fig. 11*b*, drawn from a fixed point, be imagined to change in such a way as to represent at each instant the velocity of the body as it describes its orbit, then the end P of the line will describe a curve called the *hodograph* of the orbit, and of course, the *velocity of the point* P will represent at each instant the *acceleration of the moving body.* Thus the hodograph of a

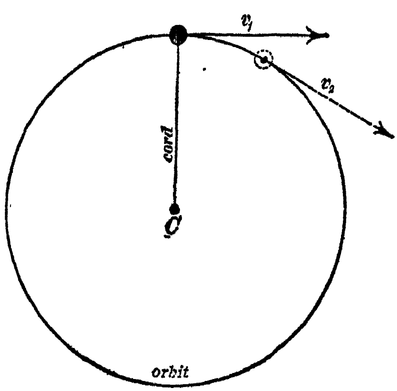


Fig. 11*a*.

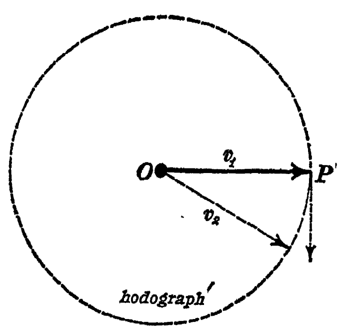


Fig. 11*b*.

tossed ball is a vertical straight line as shown in Fig. 10*b*, this is evident when we consider that a tossed ball has a constant *acceleration* vertically downwards, so that the point P, Fig. 10*b*, must move at a constant *velocity* vertically downwards. The hodograph of a ball twirled on a string is a circle as shown in Fig. 11*b*, this is evident when we consider that the magnitude of the velocity of the body in Fig. 11*a* is constant, so that the length of OP, Fig. 11*b*, which represents the velocity of the body, must also be constant. Furthermore, the velocity of the ball is always at right angles to the cord in Fig. 11*a*, and therefore the line OP, Fig. 11*b*, is always at right angles to the cord, so that the generating point P of the hodograph makes the same number of revolutions per second as the twirled ball. It is important to note also that the velocity of the point P in Fig. 11*b*, is always parallel to the cord in Fig. 11*a*, that is, the acceleration of the ball in Fig. 11*a* is at each instant in the direction of the cord.

Space variation of vectors. The simplest idea connected with the space variation of a vector is the idea of the stream line, if it may be permitted to use the terminology of fluid motion to designate a general idea. A stream line in a moving fluid is a line drawn through the fluid so as to be at each point parallel to the direction in which the fluid is moving at that point. Thus if a pail of water be rotated about a vertical axis, the stream lines are a system of concentric circles as shown in Fig. 12*a*, and Fig. 12*b* represents the approximate trend of the stream lines in the case of a jet of water issuing from a tank. Where the stream lines come close together the velocity of the water is great and where they are far apart the velocity is small.

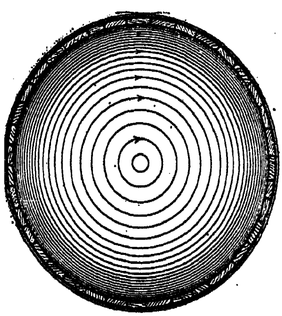


Fig. 12*a*.

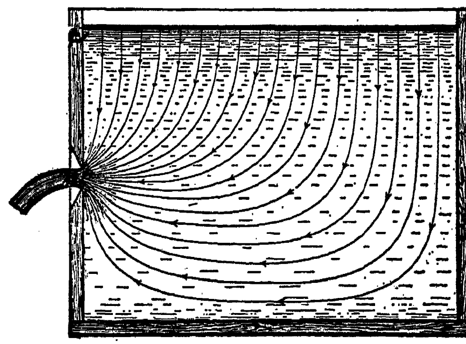


Fig. 12*b*.

The potential of a distributed vector.—In some simple cases of fluid motion it is geometrically possible to look upon the stream lines as the lines of slope of an imagined hill, the steepness of which represents the velocity of the fluid at each point both in magnitude and in direction. The height of this imagined hill at a point is called the *potential* of the fluid velocity at that point. The idea of potential is especially useful in the study of electricity and magnetism.*

PROBLEMS.

7. A table top is 10 feet long and 50 inches wide. Find its area in inch-feet, and explain the result. Ans. 500 inch-feet. One inch-foot is the area of a rectangle one foot long and one inch wide.

8. A body has a mass of 60 pounds and a volume of 2 gallons. Find its density without reducing data in any way. Ans. 30 pounds per gallon.

9. A water storage basin has an area of 2,000 acres, find the

*A very simple discussion of the theory of scalars and vectors in space, including the theory of potential, is given in Chapter VI of *Electric Waves*, by W. S. Franklin; The Macmillan Co., New York, 1909.

volume of water in acre-feet required to fill the basin to a depth of 16 feet. Explain the acre-foot as a unit of volume and find the number of gallons in one acre-foot. Ans. 32,000 acre-feet. The acre-foot is the volume of water required to fill a one-acre pond one foot deep. It is equal to 325,800 gallons.

10. A man travels at a velocity of 6 feet per second; how far does he travel in two hours? Find the result without reducing the data in any way. Explain the *foot-hour per second* as a unit of length. Ans. Distance traveled is 12 hour-feet-per-second; one hour-foot-per-second is the distance passed over in one hour by a man walking at a speed of one foot per second.

11. A man starts from a given point and walks three miles due north, then two miles northeast, then 2 miles south, and then one mile east. Show by means of a vector diagram how far, and in what direction he is from his starting point. Ans. Draw an actual diagram to scale showing the distances traveled.

12. A stream flows due south at a velocity of two miles per hour. A man rows a boat in an eastward direction at a velocity of four miles per hour. What is the actual velocity of the boat and in what direction is it moving? Ans. 4.472 miles per hour; 63° 26′ east of south.

Note. The actual velocity of the boat is the vector sum of the velocity of the boat with reference to the water (an eastward velocity of 4 miles per hour) and the velocity of the stream (a southward velocity of 2 miles per hour).

13. A stream flows due south at a velocity of two miles per hour. A man, who can row a boat at a velocity of four miles per hour, wishes to reach the opposite bank at a point due east of his starting point. In what direction must he row? Ans. 30° up stream.

Note. The actual velocity of the boat is, in this case, to be an eastward velocity, and it is equal to the vector sum of the velocity of the boat with reference to the water and the velocity of the water.

From a point O draw a line OA representing the southward velocity of the stream. From the point O draw a line OB in a due eastward direction. With the point A as a center describe a circle of which the radius represents a velocity of 4 miles per hour, and let P be the point where this circle cuts the line OB. Then the line AP represents the required direction in which the boat must be rowed.

14. An anemometer on board ship indicates a wind velocity of 28 miles per hour apparently from the northeast. The ship, however, is moving due north at a velocity of 15 miles per hour. What is the actual direction and velocity of the wind? Ans. 13° 37′ south of west; velocity of wind 20.37 miles per hour.

Note. The apparent velocity of the wind to a person on shipboard is the vector sum of the two parts, namely, (*a*) the actual velocity of the wind and (*b*) what may be called the *boat wind*, that is to say, a wind which is due to the velocity of the boat alone. This *boat wind* is equal and opposite to the velocity of the boat. In this particular problem the apparent wind of 28 miles per hour from the northeast is the vector sum of the actual wind and the boat wind of 15 miles per hour towards the south. Therefore draw a line OA representing the apparent velocity of the wind of 28 miles per hour towards the southwest. Draw another line OB representing the boat wind of 15 miles per hour towards the south. Then the line BA represents the direction and velocity of the actual wind.

15. A gun which produces a projectile-velocity of 200 feet per second is mounted aboard a car with its barrel G at right angles to the direction of motion of the car, as shown in Fig. 15*p*. The

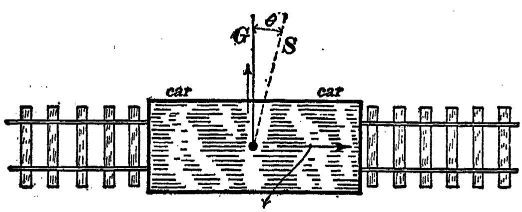


Fig. 15*p*.

car is traveling 50 feet per second. The sights are to be arranged at an angle θ to the gun barrel as shown. Find the value of θ so that the ball may hit any object which, at the instant of firing, is in the line S of the sights. Ans. $\theta = 14° \ 2' \ 10''$; actual velocity of projectile is 206 feet per second.

Note. The actual velocity of the bullet in this problem is the vector sum of the velocity of the car and 200 feet per second at right angles to the motion of the car, and the sight line must, of course, be in the actual direction in which the bullet is moving when it leaves the gun.

16. A body moves at a velocity of 20 miles per hour in a direction 20° north of east; find the northward and eastward com-

ponents of its velocity. Ans. Northward component 6.84 miles per hour; eastward component 18.8 miles per hour.

17. Find the magnitude and direction of the single force which is equivalent to the combined action of three forces A, B and C; force A being northward and equal to 200 pounds, force B being towards the north-east and equal to 150 pounds, and force C being eastwards and equal to 100 pounds. Ans. 370 pounds; 34° 31′ east of north.

18. A horse pulls on a canal boat with a force of 600 pounds-weight and the rope makes an angle of 25° with the line of the boat's keel. Find the component of the force parallel to the keel. Ans. 543.8 pounds.

19. If 500 grams of water leak out of a pail in 26 seconds, what is the average rate of leak? Ans. 19.22 grams per second.

20. A man earns $27.50 in 8½ days. What is the average rate at which he earns money? Ans. $3.235 per day.

21. A man's wages increase from $50 per month to $150 per month in the course of 3 years. What is the average rate of increase of wages? Ans. $33⅓ per month per year.

22. During 28 seconds the velocity of a train increases from zero to 12 feet per second. What is the average rate of increase of velocity? Ans. 0.429 feet per second per second.

23. A train gains a speed of 32 miles per hour in 80 seconds. Find its average acceleration in miles per hour per second. Ans. 0.4 mile per hour per second.

24. A pole 22 feet long is dragged sidewise over a field at a velocity of 8 feet per second. At what rate does the pole sweep over area? Ans. 176 square feet per second.

25. A prism has a base of 25 square centimeters, and its height is increasing at the rate of 5 centimeters per second. How fast is its volume increasing? Ans. 125 cubic centimeters per second.

26. The slope of a hill falls 60 feet in a horizontal distance of 270 feet. What is the grade? Ans. 0.222 of a foot drop for each horizontal foot of distance, or 22.2 per hundred, or 22.2 per cent.

27. One side of a brick wall is at a temperature of 0° C. and

the other side is at a temperature of 23° C. The wall is 30.5 centimeters thick. What is the average temperature gradient through the wall? Ans. 0.754 centigrade degrees per centimeter.

28. At a given point in a water pipe the water pressure is 110 pounds per square inch. Twenty-two feet from this point the pressure is 75 pounds per square inch. What is the average pressure gradient along the pipe? Ans. 1.59 pounds per square inch per foot.

CHAPTER III.

SIMPLE STATICS.*

19. Balanced force actions. When a body remains at rest, or when a body continues to move with uniform velocity along a straight line, or when a symmetrical body like a wheel continues to rotate at uniform speed about a fixed axis, the forces which act on the body are balanced. A very brief discussion of this matter is given on pages 5 to 7. The study of balanced forces or, as it is sometimes expressed, the study of forces in equilibrium, constitutes the science of statics. The science of statics is indeed the study of equilibrium in its widest sense, including the equilibrium of the forces which act on the parts of a distorted body (the statics of elasticity), and the equilibrium of the forces which act upon the parts of a fluid (hydrostatics); but these branches of statics are treated in subsequent chapters. The present chapter deals only with sets of forces which do not produce translatory motion or rotatory motion.†

Every one knows that a single force acting on a body may cause both translation and rotation. Thus a boat, which is pushed away from a landing, generally turns more or less as it moves away. In this case, however, it may be the force action of the water on the boat that causes the turning; but a sidewise push on the bow of the boat certainly produces both translation and rotation independently of the force action of the water.

From the fact that a single force can produce both translation and rotation, it may seem as though it would be impossible to consider separately the two effects of a force, namely (*a*) the tendency to produce translatory motion, and (*b*) the tendency to

*One of the best treatises on statics is Part II of Alexander Ziwet's *Theoretical Mechanics* (The Macmillan Co.). A very complete treatment is G. M. Minchin's *Treatise on Statics* in two volumes. (Clarendon Press.)

†See Art. 29 for definitions of translatory motion and rotatory motion.

produce rotatory motion; but every one knows that forces can produce translation without producing rotation, and that forces can produce rotation without producing translation. Thus, a table can be moved to one side without turning, or a table can be turned without being moved to one side. The fact is that the two tendencies *a* and *b* must be considered separately. The tendency of a force to produce translatory motion and the tendency of a force to produce rotatory motion constitute two distinct types of force action as explained in Art. 29.

The tendency of a force to produce translatory motion depends only upon the direction and magnitude of the force.*

The tendency of a force to produce rotatory motion about a given axis is called the *torque action* of the force about that axis or the *moment of the force* about that axis, and it is equal† to the product of the force and the perpendicular distance from the

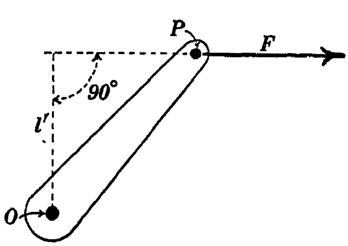


Fig. 13*a*.

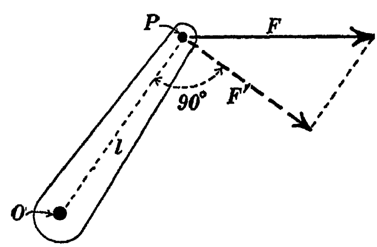


Fig. 13*b*.

axis to the line of action of the force. Thus the torque action of the force F about the axis O in Fig. 13*a* is equal to the product Fl'. The torque action of the force F about the point O in Fig. 13*a* is also equal to the product $F'l$ as shown in Fig. 13*b*.

*This statement appears on its face to be contrary to experience as follows: Every one knows that a force applied at the middle of a bureau drawer may pull the drawer out, whereas the same force applied at one end will not pull the drawer out. When the outward pull is applied at the end of the drawer, however, it causes the drawer to bind against the guides *thus bringing into action a new set of forces* which do not exist when the outward force is applied at the middle of the drawer.

†The simplest argument which leads to an idea of the value of a torque is the argument which is given in Art. 60.

20. First condition of equilibrium. The condition that must be satisfied in order that a number of forces may have no tendency to produce translatory motion is called the first condition of equilibrium and it may be stated in two ways as follows:

(*a*) In order that a number of forces may have no tendency to produce translatory motion, it is necessary that what is called the vector sum* of the forces be equal to zero, that is, the respective forces must be parallel and proportional to the sides of a closed polygon and in the directions in which the sides of the polygon would be passed over in going round the polygon. This statement of the first condition of equilibrium leads directly to the graphical method of solving problems in statics.

(*b*) Imagine each of the forces acting upon a body to be resolved into rectangular components (x, y and z-components). Then the sum of all the x-components must be equal to zero, the sum of all the y-components must be equal to zero, and the sum of all the z-components must be equal to zero. In applying this condition of equilibrium it is frequently convenient to pick out all of the positive x-components (to the right) and all the negative x-components (to the left) and place the sum of the forces to the right equal to the sum of the forces to the left; and to proceed in a similar way with respect to the y and z-components.

Examples of the application of the first condition of equilibrium. Problems in statics which involve no question of rotation are solved by applying the first condition of equilibrium. There are four cases as follows: (*a*) When everything is given but the magnitude and direction of one force, (*b*) when everything is given but the magnitude of one force and the direction of another, (*c*) when everything is given but the magnitudes of two forces, and (*d*) when everything is given but the directions of two forces.

Solution of case (*a*). Starting from a point O in Fig. 14*a*, draw a line representing the known direction and magnitude of force No. 1. From the end of this line draw a line representing

*See Art. 14.

the known direction and magnitude of force No. 2. From the end of this line draw a line representing the known direction and magnitude of force No. 3, and so on. Let P be the point

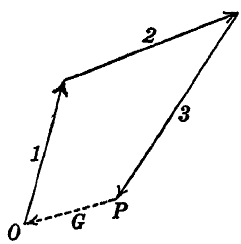


Fig. 14*a*.

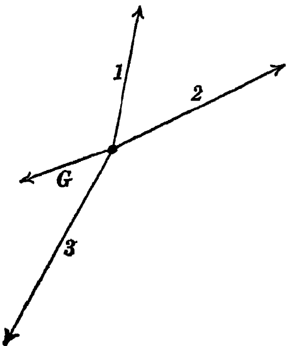


Fig. 14*b*.

ultimately so reached. Then the line PO represents the unknown force in magnitude and direction. Figure 14*b* shows the forces as they appear when acting on the body.

Solution of case (*b*). Starting from a point O, Fig. 15*a*, draw a line representing the known direction and magnitude of force No. 1. From the end of this line draw a line representing the known direction and magnitude of force No. 2, and so on. Let

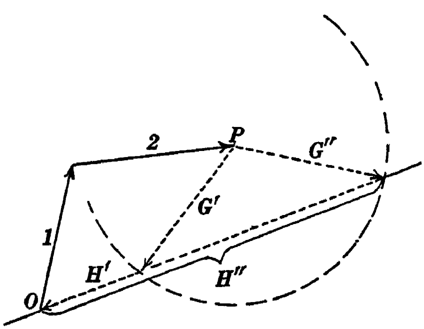


Fig. 15*a*.

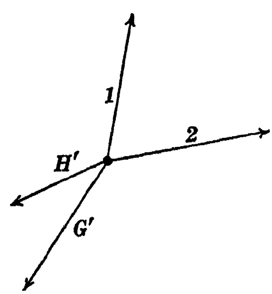


Fig. 15*b*.

P be the point ultimately so reached. From P as a center describe a circle of which the radius represents the known value of force G, and from O draw a line in the known direction of force H.

The problem has two solutions; the forces corresponding to one solution are 1, 2, G' and H'; and the forces corresponding to the other solution are 1, 2, G'' and H''. Figure 15*b* shows the forces 1, 2, G' and H' as they appear when acting on the body.

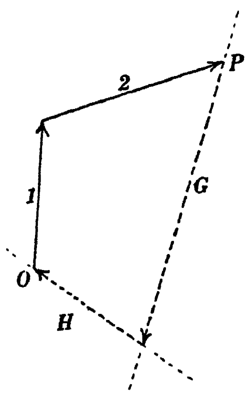


Fig. 16*a*.

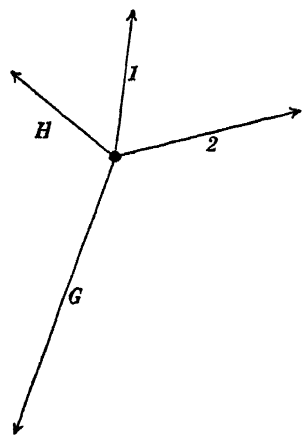


Fig. 16*b*.

Solution of case (*c*). Starting from a point O, Fig. 16*a*, draw a line representing the known direction and magnitude of force No. 1. From the end of this line draw a line representing the known direction and magnitude of force No. 2, and so on. Let

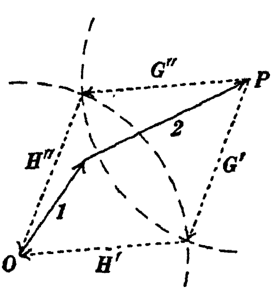


Fig. 17*a*.

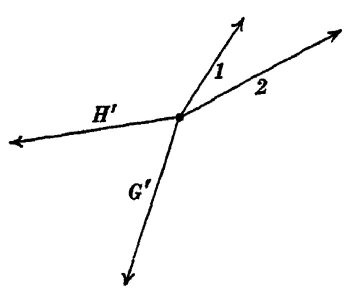


Fig. 17*b*.

P be the point ultimately so reached. Through the point P draw a line in the known direction of force G and through the point O draw a line in the known direction of force H. The lines G and H in Fig. 16*a* represent the required magnitudes of

the two forces. Figure 16*b* shows the forces 1, 2, G and H as they appear when acting on the body.

Solution of case (*d*). Starting from a point O, Fig. 17*a*, draw a line representing the known direction and magnitude of force No. 1. From the end of this line draw a line representing the known direction and magnitude of force No. 2, and so on. Let P be the point ultimately so reached. With P as a center describe a circle of which the radius represents the known magnitude of force G, and with O as a center describe a circle of which the radius represents the known magnitude of force H. This problem has two solutions; the forces corresponding to one solution are represented by the lines 1, 2, G' and H'; and the forces corresponding to the other solution are represented by the lines 1, 2, G'' and H''. Figure 17*b* shows the forces 1, 2, G' and H' as they appear when acting on the body.

Remark. The above discussion of the four cases, (*a*) (*b*) (*c*) and (*d*), is based upon the first form of statement of the first condition of equilibrium. To show that the second form of statement leads to the same result let us consider case (*a*) in which the forces 1, 2, 3 are given and the force G is to be found. Figure 18 shows the x-components of the forces 1, 2, 3 and G, and it is evident from the figure that $X_1 + X_2 = X_3 + X_G$ from which X_G may be calculated. In the same way the y-component of the unknown force G may be calculated.

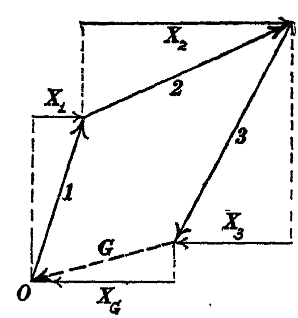


FIG. 18.

The identity of the first and second statements of the first condition of equilibrium may be appreciated if one considers that the sum of the x-components of the sides of a closed polygon is equal to zero, x-components to the right being considered as positive and x-components to the left being considered as negative.

21. Second condition of equilibrium. In order that a number of forces may have no tendency to turn a body about a chosen axis it is necessary that the sum of the torque actions of the forces about that axis be equal to zero; torques tending to turn the body in one direction being considered as positive and torques tending to turn the body in the opposite direction being considered as negative. The chosen axis (or point) is called the *origin of moments*. In applying the second condition of equilibrium it is usually convenient to pick out all of the torques which tend to turn the body in a clock-wise direction, and all of the torques which tend to turn the body in a counter-clock-wise direction, and to place the sum of the clock-wise torques equal to the sum of the counter-clock-wise torques.

Examples of the application of the second condition of equilibrium. Problems in statics which refer to a body mounted upon a *fixed axis* involve no question of translation, and they are solved by using the second condition of equilibrium. Thus one point of a lever, *the fulcrum*, is fixed in position, and the equilibrium of the lever depends upon balanced torque actions about the fulcrum.

First example. Consider the two forces A and B which act upon a lever as shown in Fig. 19*a*. The torque action of the force A about the fulcrum O is equal to Aa, and the torque action of the force B about the fulcrum O is equal to Bb. These two

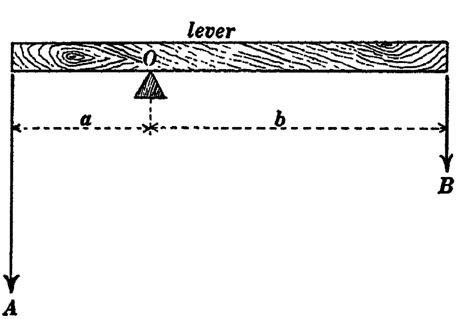


Fig. 19*a*.

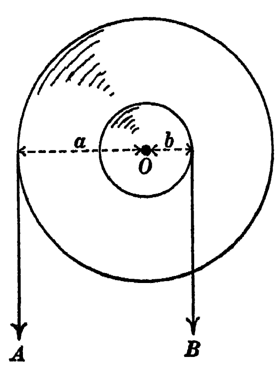


Fig. 19*b*.

torques are opposite in direction and therefore to balance each other they must be equal in value; or, in other words, we must have

$$Aa = Bb$$

Second example. Figure 19*b* represents a wheel and axle, the radius of the wheel being a and the radius of the axle being b. The torque action of the force A about the axis O is equal to Aa, and the torque action of the force B about the axis O is equal to Bb. These two torques are opposite in direction and therefore to balance each other they must be equal in value; or, in other words, we must have

$$Aa = Bb$$

22. Examples of the application of first and second conditions of equilibrium. *First example.* Figure 20*a* represents a bell-

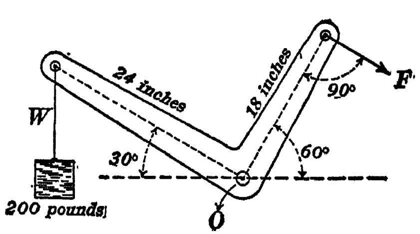


Fig. 20*a*.

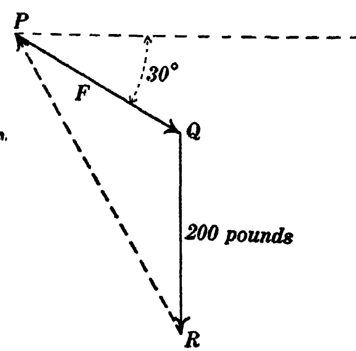


Fig. 20*b*.

crank lever of which the fulcrum is at the point O. It is required to find the value of the force F, and to find the value and direction of the force which acts upon lever at O. Applying the second condition of equilibrium we have

$$F \times 18 \text{ inches} = 200 \text{ pounds} \times 24 \text{ inches} \times \cos 30°$$

The first member of this equation is the torque action of the force F about O, and the second member is the torque action of the

200-pound weight about O. This equation determines the value of the force F.

To determine the force which acts upon the lever at O, draw a line PQ in Fig. 20*b* representing the known value and direction of the force F, and from the end of this line draw the line QR representing the knòwn force W in Fig. 20*a*. The line RP then represents in magnitude and direction the force which acts upon the lever at O.

Second example. A table drawer 36 inches in breadth and 18 inches in depth (front to back) is pulled by a force F applied at a distance x from one corner as shown in Fig. 21. The drawer binds at the two corners p and q, and it is required to find the

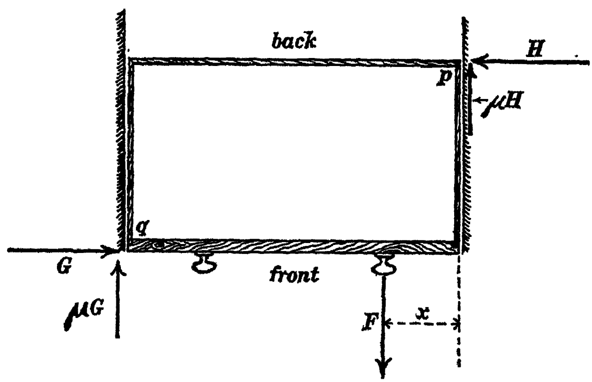


Fig. 21.

smallest value of x for which the drawer can be pulled out by the force F, the coefficient of friction* between drawer and guides being 0.7.

The guides exert forces upon the drawer at the two corners p and q, and when the drawer is on the point of sliding the force exerted on the drawer at p has the two components H and μH as shown, and the force exerted on the corner at q has the two components G and μG as shown, μ being the coefficient of friction between drawer and guides.

From the first condition of equilibrium the sum of all the

*See Arts. 51 and 52 for a discussion of friction.

forces to the right must be equal to the sum of all the forces to the left, that is

$$G = H \tag{i}$$

and also the sum of all the downward forces must be equal to the sum of all the upward forces in the figure; that is

$$F = \mu G + \mu H \tag{ii}$$

Choosing the origin of moments at the point q, we have

$$\begin{array}{ccc} \text{Right-handed torques.} & & \text{Left-handed torques.} \\ F(b - x) & = & b\mu H + aH \end{array} \tag{iii}$$

in which a is the depth of the drawer front to back, and b is its width.

All three unknown forces F, G and H may be eliminated from these equations and the value of x determined in terms of b, a and μ.

23. Pure torque. A number of forces which act on a body may have no tendency to produce translatory motion and still have a tendency to produce rotation, or, in other words, a number of

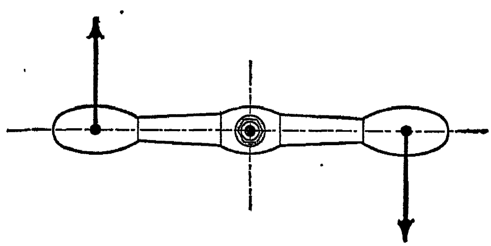


Fig. 22.

forces may satisfy the first condition of equilibrium and not satisfy the second condition. Such a combination of forces constitutes a *pure torque* and the total torque action is the same about any point whatever. For example, the two equal and opposite forces which are exerted on the handle of an auger, as shown in Fig. 22, constitute a pure torque. Such a pair of forces is sometimes called a *couple.*

24. Three forces in a plane must intersect at a point if they are in equilibrium, except when the forces are parallel. Choose the origin of moments at the intersection of the lines of action of two of the forces. The torque actions of these two forces about the origin is then equal to zero, and if the third force does not pass through the origin it will have an unbalanced torque about the origin and the forces will not be in equilibrium.

When a problem in statics refers to the equilibrium of three forces the second condition of equilibrium is completely accounted for, insofar as it has a bearing on the problem, by considering that the lines of action of the three forces must intersect at a point.

Example. Extend the lines of action of the forces W and F backwards in Fig. 20*a* until they intersect at a point T (point T not shown in the figure). The line of action of the force which is exerted on the lever at O is the line OT. Having thus determined the direction of the force at O, and knowing the direction of the force at F in Fig. 20*a*, the problem of finding the value of the force at O is reduced to case (*c*) in Art. 20.

25. Resultant of a number of forces. Any number of forces (not in equilibrium) which act on a body are together equivalent to a single force which is called their resultant; except when the forces constitute a pure torque.

Proof. Given a number of forces in equilibrium. If one of these forces (F) is omitted, the combined action of the others must be a force equal and opposite to F and having the same line of action as F. The exception is also evident since by omitting one of a set of forces in equilibrium, the others cannot constitute a pure torque.

The magnitude and direction of the resultant of a number of given forces is found by taking the vector sum of the forces. The point of application of this resultant is a point about which the given forces have no torque action.

First example. Three well matched horses are hitched to an evener EE' as shown in Fig. 23*a*. Horse 1 exerts a certain force

A at the point E, and horses 2 and 3 exert at the point E' a force B which is equal to $2A$. The forces A and B are parallel, and the value of their resultant R is therefore equal to their numerical sum. The point of application of the resultant R is the point O about which the torque actions of A and B balance each other. Therefore the forces exerted by the three horses are together equivalent to a single force R acting at the point O.

Second example. Figure 23*b* shows two given forces A and B acting at the ends of a bar. The resultant of the two forces is

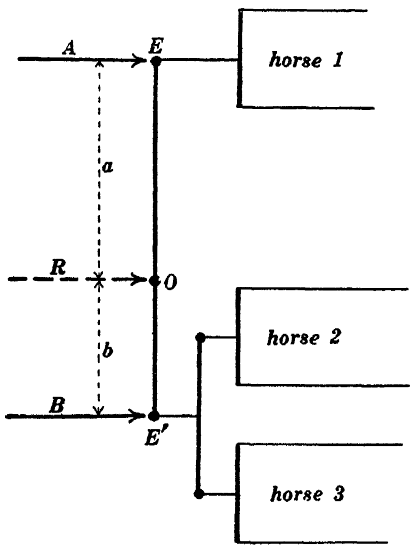


Fig. 23*a*.

represented in direction and magnitude by the diagonal R', and the point of application O of the resultant may be at any point on the line $PR'R$. The construction of the figure is sufficiently evident without further explanation.

26. Center of gravity.* The center of gravity of a body is the point of application of the total force with which the earth pulls the body. Thus the total force with which the earth pulls a uniform bar may be thought of as applied at the center of the bar.

*Considering the forces of gravity on the various parts of a body to be accurately parallel, the *center of gravity* of a body is the same thing as the *center of mass* of the bodv.

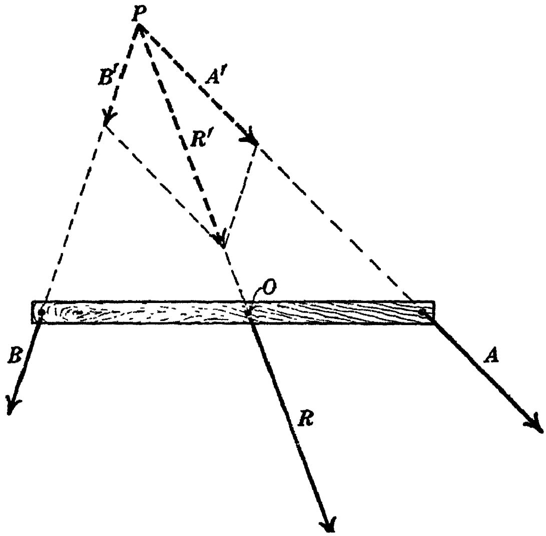


Fig. 23*b*.

Example. A uniform board 16 feet long and weighing 25 pounds is held in a horizontal position with one end resting on a table, and a point 6 feet from the other end resting on the hand. It is required to find the forces T and H with which the table and hand respectively push upwards on the board. From the first condition of equilibrium we have

$$T + H = 25 \text{ pounds} \qquad \text{(i)}$$

Choosing the origin of moments at the center of the board, the second condition of equilibrium gives

$$T \times 8 \text{ feet} = H \times 2 \text{ feet} \qquad \text{(ii)}$$

The first member of this equation is the torque action of the upward push of the table about the origin of moments (center of the board), and the second member is the opposite torque action of the upward push of the hand about the center of the board. These two equations enable the calculation of T and H.

27. Action and reaction.* A rope pulls on a canal boat and the canal boat pulls backwards on the rope with an equal force.

*This matter is discussed very briefly on page 5.

A body resting on a table pushes downwards on the table and the table pushes upwards on the body with an equal force. *The mutual force action between the two bodies A and B always consists of two equal and opposite forces one of which acts on A and the other on B.* **It is of the utmost importance in the consideration of any problem in statics to think of the forces which act on a certain body, because a system of forces must act upon one body to constitute a system of forces in equilibrium;** to think of the forces with which the body under consideration acts on other bodies leads to confusion. A body rests upon a table; the earth pulls downward on the body and of course the body pushes downward on the table with an equal force. These two forces do not, however, constitute a system in equilibrium, because one of them acts on the body and the other acts on the table; the pull of the earth on the body is balanced by the upward push of the table on the body.

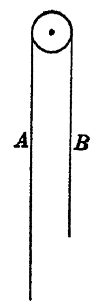


Fig. 24*a*.

First example. A man weighing 180 pounds is seated in a stirrup which is suspended by the rope A as shown in Fig. 24*a*. It is required to find the force with which the man must pull downwards on rope B to lift himself. Let F be the downward pull of the man on rope B. The reaction is an upward pull of rope B on the man equal to F. At the same time rope A pulls upwards on the man with a force F, so that the total upward pull on the man is $2F$; therefore F is equal to 90 pounds.

Second example. When a problem in statics refers to a mechanism consisting of several pieces, like a train of gears, the principle of equality of action and reaction must be used in discussing the equilibrium of the successive parts of the mechanism. Thus Fig. 24*b* shows two levers LL and $L'L'$. A force F is exerted upon the lever L, the other end of this lever exerts a force H upon the lever L', and the other end of the lever L' supports a weight W. It is required to find the weight W, having given the value of the force F and the two ratios A/a and B/b. The

torque action of the force F about the fulcrum O is balanced by the torque action of the force G with which the lever L' acts upon the lever L. Therefore $G = F \times A/a$. Having thus calculated the force G, the force H is known (equal and opposite to G), and the value of W is given by the equation $W = H \times B/b$.

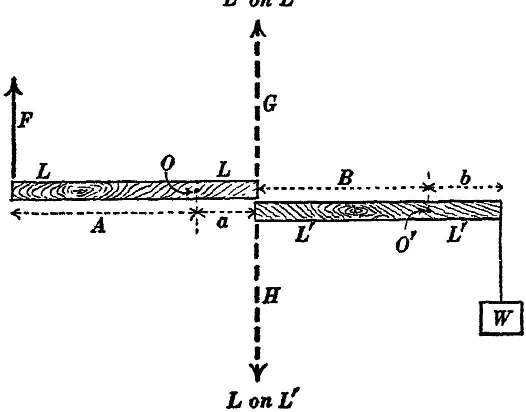


Fig. 24*b*.

The principle of virtual work. If the forces which act on a body are in equilibrium, the total work done by the forces during any small movement of the body would be equal to zero. This is called the principle of virtual work. It is discussed and illustrated in Chapter V.

D'Alembert's principle. The ease with which the relationship between a number of forces in equilibrium can be shown, especially by the use of graphical methods, makes it desirable to extend the idea of balanced forces to the subject of dynamics. The principle on which this can be done was first enunciated by D'Alembert. The following is a statement of D'Alembert's principle as applied to a particular case: A part of a mechanism moves in a prescribed manner under the combined action of a number of forces; if *fictitious forces* which are equal and opposite to the forces required to produce the known accelerations be introduced into the system, then the system of forces (including the fictitious forces) will be in equilibrium. D'Alembert's principle is exemplified by problems 54 and 56 on pages 112 and 113.

PROBLEMS.*

29. The sail SS, Fig. 29*p* of a sailboat is assumed to be a smooth plane, and the force exerted on the sail by the wind, neglecting

*See *Problems in Statics*, by Franklin and MacNutt; a collection of problems in statics with very complete notes as to methods of solution. Published in pamphlet form by the Macmillan Company.

the slight tangential force (parallel to plane of sail) exerted by the wind as it slides or glances along the sail, is at right angles to the plane of the sail SS. This force is equal to 200 pounds. Find its components parallel to and at right angles to the center-board BB. Ans. 100 pounds parallel to center-board; 173.2 pounds at right angles to center-board.

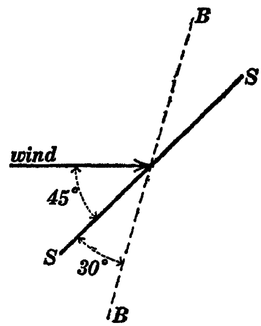


Fig. 29*p*.

Note. The component of the force at right angles to the center-board BB is counteracted by the pressure of the water against the center-board, and this component produces no perceptible motion of the boat because of the large area of the center-board. The component of the force which is parallel to the center-board BB is the force which propels the boat forwards.

30. A ball weighing 100 pounds is suspended by a rope A. The ball is pulled to one side by a second rope B, the direction of rope B is 30° below the horizontal and the tension of rope B is 60 pounds. Find the direction and tension of rope A. In answer to this problem make a diagram in which three lines, drawn outwards from a point, represent the magnitudes and directions of the three forces acting on the ball. Ans. 140 pounds; 68° 13′ above the horizontal.

31. A ball weighing 100 pounds is suspended by a rope A. The ball is pulled to one side by a second rope B which is tied to it. Find the two possible directions of rope B for which its tension is 60 pounds, and find the tension of rope A, the angle between rope A and the vertical being 30°. This problem has two answers. For each answer make a diagram in which three lines, drawn outwards from a point, represent the magnitudes and directions of the three forces acting on the ball. Ans. 63° 31′ above the horizontal; 53.45 pounds. 3° 45′ above the horizontal; 119.75 pounds.

32. A ball weighing 2 pounds hangs by a string and is pushed to one side by a strong wind (horizontal) so as to give an angle of 30° between the string and the vertical line. Find the tension

of the string and the force exerted on the ball by the wind. In answer to this problem make a diagram in which three lines, drawn outwards from a point, represent the magnitudes and directions of the three forces acting on the ball. Ans. Tension in string 2.31 pounds; force of wind 1.15 pounds.

33. A ball weighing 100 pounds is suspended by a rope A and the ball is pulled to one side by a second rope B which is tied to it. Find the directions of ropes A and B in order that the tension of A may be 80 pounds and the tension of B 60 pounds. Make a diagram in which three lines, drawn outwards from a point, represent the three forces acting on the ball. Ans. 53° 8′ above the horizontal.

34. A rope 15 feet long stretched between two supports SS carries a 1000-pound weight as shown in Fig. 34*p*. Find the tension in each part of the rope, and find the vertical force on each

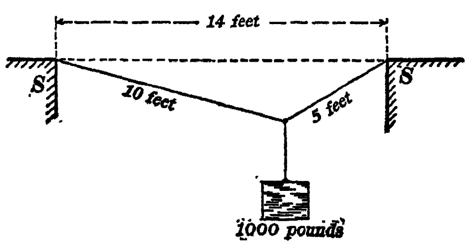


Fig. 34*p*.

support, neglecting weight of rope. Ans. 1372 pounds in 5-foot section; 1230 pounds in 10-foot section; 691 pounds at right-hand support; 309 pounds at other support.

35. A simple bridge truss consists of two struts and a tie rod, as shown in Fig. 35*p*. A weight W of 2000 pounds hangs from the point P. Find the compression in each strut, the vertical pressure on each abutment, and the tension in the tie rod, neglecting weights of the parts of the truss. Ans. Compression in 10-foot strut 1720 pounds. Compression in 13-foot strut 1450 pounds. Tension in tie rod 1220 pounds. Vertical force at R, 1213 pounds. Vertical force at Q, 787 pounds.

36. A forty-foot beam, arranged as shown in Fig. 36*p*, supports a weight of 2000 pounds. Find the pull of the rope and the

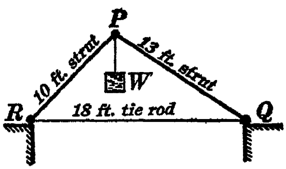


Fig. 35*p*.

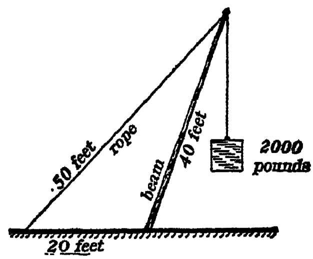


Fig. 36*p*.

thrust of the beam. Ans. Tension in rope 1645 pounds. Compression in beam 3420 pounds.

37. The two links of a toggle joint are each 10 inches long between centers of end pins, and the center of the middle pin P

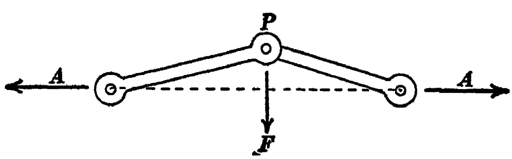


Fig. 37*p*.

in Fig. 37*p* is 0.5 inch from the center line of the end pins (the dotted line in Fig. 37*p*). Find the horizontal components AA of the forces exerted on the end pins when the force F is 50 pounds, ignoring friction. Ans. 499.4 pounds.

Note. To neglect the friction is to consider the forces exerted at P to be parallel to the respective links of the toggle.

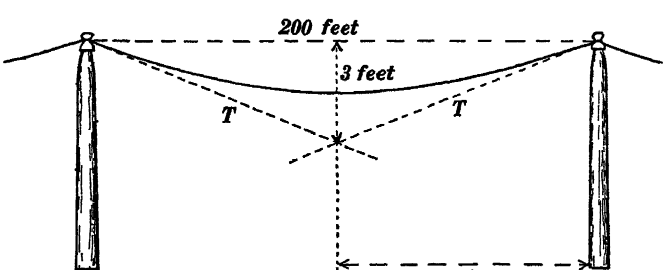


Fig. 38*p*.

38. The span of wire on a telegraph line is 200 feet long, and the sag of the line is such that the two lines TT, Fig. 38*p*, intersect at a point three feet below the level of the insulators on the poles, the two lines TT being tangent to the wire at the insulators. The total weight of the wire in the span is 15 pounds. Find the tension of the wire at the insulators, find the tension of the wire at the center of the span, and find the total downward force upon an insulator. Ans. Tension of wire at insulators 250.11 pounds; tension at center 250 pounds. Downward force at insulator 7.5 pounds.

Note. To solve this problem consider one-half of the span of wire A as a rigid body in equilibrium. The force with which the insulator pulls on this body is equal and opposite to the tension of the wire at the insulator. To apply the second form (b) of the first condition of equilibrium, consider that the force pulling to the left on the half-span A is the tension of the wire at the center of the span, and that the force pulling to the right on the half-span of wire is the horizontal component of the tension of the wire at the insulator; and consider that the force pulling the half-span of wire upwards is the vertical component of the tension of the wire at the insulator and that the force pulling the half-span of wire downwards is the weight of the wire in the half-span.

39. The cable of a suspension bridge is anchored to a block of masonry as shown in Fig. 39*p*. The block is in the form of a cube the length of edge of which is 40 feet, the cable being assumed to pass over the sharp edge of the cube at C. The weight of the masonry is 160 pounds per cubic foot. Required the tension in the cable which will just suffice to turn the block of masonry over. Ans. 2090 tons.

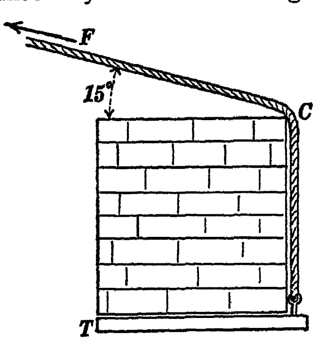


Fig. 39*p*.

Note. The pull of gravity may be thought of as applied at the center of the block of masonry, and the problem may be solved by equating torques about the point T. If it is desired to find the force which the earth exerts on the pier at T, one may proceed as follows: (1) Draw the line of action of the force with which the earth pulls on the pier; this is a vertical line passing through the center of the pier; let P be the point where this line intersects FC; then TP is the line of action of the force exerted on the pier at T, according to Art. 24. (2) Having

determined the direction of the force at T we have everything given but the magnitude of two forces and therefore the problem falls under case c, Art. 20.

40. Find the force F in Fig. 40*p* required to draw a 200-pound block up an inclined plane of the dimensions shown, the coefficient of friction between block and plane being 0.2. Ans. 95 pounds.

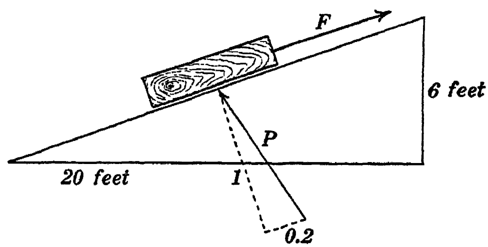


Fig. 40*p*.

Note. The block is in equilibrium under the action of three forces; (1) the downward pull of the earth of 200 pounds, (2) the unknown force F (of which the direction is known), and (3) the unknown force exerted on the block by the plane. This force is in the direction of the line P in Fig. 40*p*. See Art. 52 for a discussion of coefficient of friction.

41. A wedge of which the shape is indicated in Fig. 41*p* is pushed between two blocks A and B with a force F of 5000 pounds. The coefficient of friction between the wedge and the blocks is 0.2. Find the components at right angles to F of the forces with which the wedge pushes on A and B. Ans. 8676 pounds.

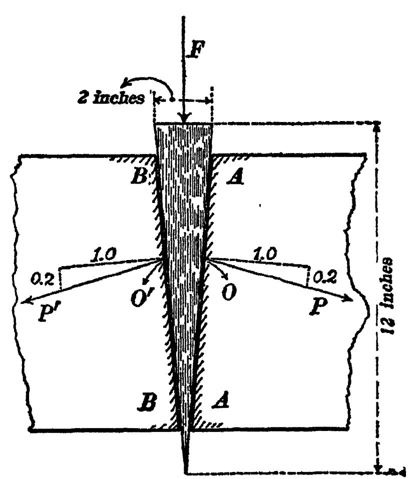


Fig. 41*p*.

Note. The lines OP and $O'P'$ in Fig. 41*p* represent the forces with which the wedge pushes against the blocks AA and BB respectively. The blocks push against the wedge with equal and opposite forces. Therefore, of the three forces which act on the wedge, one is given in magnitude and direction, the directions of the other two are known, and the magnitudes of the other two are to be found. This problem, therefore, falls under case (c) in Art. 20.

42. The diameters of the two drums D and d in Fig. 42p are 4 feet and 1 foot respectively, and the diameters of the pitch circles of the two cog-wheels W and w are 5 feet and 10 inches, respectively. Find the weight A. Ans. 4800 pounds.

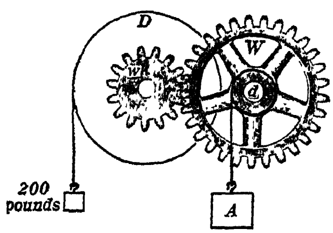


Fig. 42p.

Note. When the point of contact of two cogs on W and w is on the line of the centers of the two wheels, the point of contact is at certain distances r and R respectively from the centers, where r and R are the radii of what are called the *pitch circles* of the two cog-wheels. The cog-wheels may be thought of as *friction drums* of which the diameters are equal to the diameters of the pitch circles of the respective cog-wheels. What force balances the torque action of the 200-pound weight on the body Dw, what exerts this force, upon what is the force exerted, what is the direction of the force, and at what distance from the center of w is the line of action of this force? What force balances the torque action of the weight A on the drum d, what exerts this force, upon what is the force exerted, what is the direction of the force, and how far is the line of action of the force from the center of d?

43. A uniform stick 6 feet long and weighing 10 pounds has three weights hung upon it as shown in Fig. 43p. Find the distance x from the end of the stick to the point where the single force F must be applied to produce equilibrium. Ans. 3.18 feet.

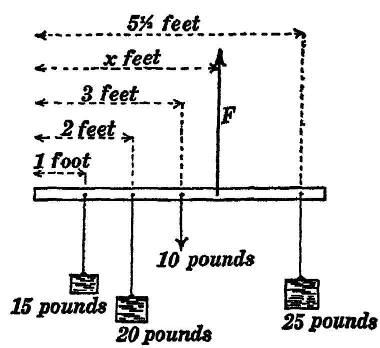


Fig. 43p.

44. The steam in an engine cylinder pushes on the piston with a force of 12,000 pounds-weight. The positions and lengths of connecting rod and crank are shown in Fig. 44p. Find the force with which the cross-head pushes sidewise against the guide, the thrust of the connecting rod, and the torque in pound-feet exerted on the crank-shaft, neglecting friction throughout. Ans. Side-thrust 1257 pounds; thrust in connecting rod 12065 pounds; torque 5000 pound-feet.

45. A ladder 16 feet long and weighing 100 pounds has its center of gravity 7 feet from its lower end which stands on a floor at a distance of 4 feet from a vertical wall against which

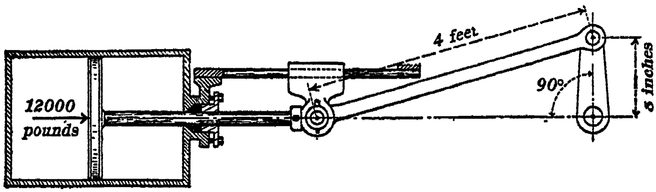


Fig. 44*p*.

the ladder rests, as shown in Fig. 45*p*. Assuming the force *a* with which the wall pushes on the ladder to be horizontal, find the magnitude of *a* and the direction and magnitude of the force with which the ladder pushes against the floor. Ans. Force *a* 11.2 pounds; 84° with floor; 100.6 pounds.

46. An elevator car with its load weighs 1500 pounds and the center of gravity of the whole is one foot off center, as shown in Fig. 46*p*. The coefficient of friction against the guides is 0.1.

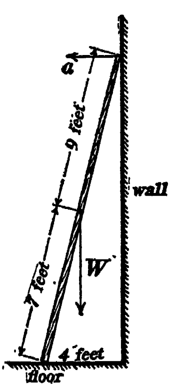


Fig. 45*p*.

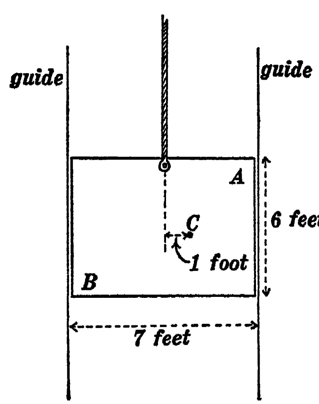


Fig. 46*p*.

Find the tension in the cable necessary to draw the car upwards at constant speed. Ans. 1550 pounds.

Note. This problem is solved in a manner exactly similar to the problem of the table drawer which is discussed on page 58.

47. Given a tackle block arranged as shown in Fig. 47p. Find the weight W which can be lifted by a force F equal to 150 pounds-weight, neglecting friction. Ans. 600 pounds.

Fig. 47p.

Note. The simplest argument of this problem is as follows: The tension of the rope is everywhere equal to 150 pounds if friction is negligible. Therefore, the four strands of rope which lead to the lower block exert a total lifting force of 600 pounds.

This problem can be handled by means of the principle of virtual work. Let d be the distance through which the force F has pulled the rope. Then $d/4$ is the distance that the weight W has been lifted, so that Fd is the work done by the force F and $Wd/4$ is the work which has been expended in lifting the weight. Therefore, ignoring friction, we have $Wd/4 = Fd$, or $W = 4F$.

48. The conical end of a vertical shaft rests in a conical seat. The total weight of the shaft and the attached wheel is 2000 pounds, and the total area of contact surface between the conical end of the shaft and the conical seat is 10 square inches. The semi-angle of the cone is 20°. Find the pressure p (force per unit area) between the conical end of the shaft and the conical seat, assuming the force to be everywhere at right angles to the surface of contact, and assuming p to have the same value over the entire surface. Ans. 585 pounds per square inch.

Note. Problems in statics in three dimensions (that is, where all the forces under consideration are not in one plane) are usually quite complicated; fortunately such problems are not frequently met with in engineering practice.

To solve this problem, consider the force p, which acts upon each unit area of the conical surface of the end of the shaft. The upward component of this force is $p \sin 20°$, and each unit of area of the conical end of the shaft contributes this same amount of upward force, so that the total upward force is 10 times $p \sin 20°$.

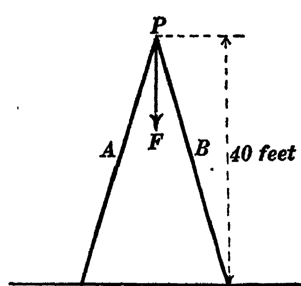


Fig. 49p.

49. The beam in problem 36 consists of two legs A and B as shown in Fig. 49p, and the edgewise view of this double beam is as shown in Fig. 36p. Find the compression in each of the

two legs A and B, using the data of problem 36. Ans. 1725 pounds.

Note. This is a problem in statics in three dimensions, but it can be easily reduced to two problems each in two dimensions. The answer to problem 36 gives the downward force F in Fig. 49*p* in the plane of the two legs A and B, and the compression in each of the legs is then easily found by considering the three forces which act upon a particle at P in Fig. 49*p*, namely, the force F and the thrusts of the two beams A and B.

50. Fig. 50*p* represents a top view of a telegraph pole P at a point where the telegraph line turns a right angle, the pole P being guyed by a wire which is inclined at an angle of 45° to the horizontal. The horizontal component of the tension of each telegraph wire is 250 pounds. Find the tension in the guy wire. Ans. 500 pounds.

Note. This is a problem in statics in three dimensions, but it can be easily reduced to two problems each in two dimensions. Find the horizontal component of the tension of the guy wire by considering that it must be equal and opposite

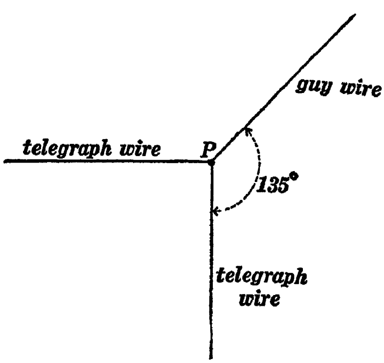


Fig. 50*p*.

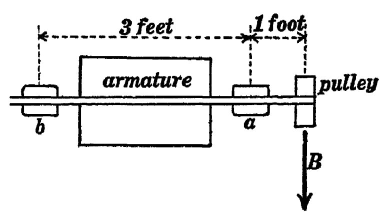


Fig. 51*p*.

to the resultant of the horizontal pulls of the two telegraph wires. Then imagine a vertical plane drawn through the guy wire. The horizontal component of the tension of the guy wire may be represented by a horizontal line in this plane, and the actual resultant tension of the guy wire will be represented by a line inclined 45° downwards.

51. Fig. 51*p* represents the armature shaft of an electric motor, the motor bearings being three feet apart, and the center of the pulley being one foot distant from the center of the adjoining bearing. The pull of the belt is a horizontal force B of 120

pounds, and the weight of the armature is a downward force of 150 pounds. Find the direction and magnitude of the force with which each bearing pushes against the armature shaft. Ans. At bearing *a* 177 pounds, 25° above horizontal. At bearing *b* 85 pounds, 62° above horizontal.

Note. This is a problem in statics in three dimensions, which can be easily reduced to four problems, each in two dimensions. Consider the force *B*, and find the two horizontal forces which must be exerted upon the shaft by the bearings to balance this force; then find the vertical forces with which the bearings must act upon the shaft to balance the downward pull of the earth on the armature; and then find the resultant of the vertical and horizontal forces exerted by each bearing.

CHAPTER IV.

DYNAMICS.* TRANSLATORY MOTION.

28. Force and its effects. When one pushes or pulls on an object one is said to exert a force on the object.† Thus the pull of a horse on a wagon is called a force. It does not, however, require an active agent‡ like a horse or an engine to exert a force. Thus a weight lying on a table exerts a downward force on the table, a string stretched between two pegs exerts a force on each peg.

The effects of a force are extremely varied. Thus to pull on a body may set the body in motion; to pull on a rubber band stretches it; to place a load on a long beam bends or breaks it; a coin is heated when force is exerted upon it in rubbing it on a board; when steam is compressed it is condensed into water; when ice is compressed a portion of the ice melts; etc. Indeed nearly every physical phenomenon involves force action of one kind or another.

A force can be measured only in terms of its effects, and the effect which can be most easily used for the measurement of force is the effect the force has in distorting a body, as, for ex-

*One of the best treatises on Mechanics for the student is *An Elementary Treatise on Theoretical Mechanics*, by Alexander Ziwet, The Macmillan Company, 1893. In keeping this work within reasonable bounds the author has excluded the more advanced parts of the subject. The book, however, gives references for the use of those who may desire to pursue the subject further.

†This matter is discussed very briefly on page 5. There has been a great deal of discussion, ever since the time of Sir Isaac Newton, concerning the nature of force. Perhaps the most significant discussion of this matter is that which is given in Appendix B (pages 268–288) of *Aether and Matter* by Sir Joseph Larmor, Cambridge University Press, 1909. This discussion of Larmor's is rather difficult to follow. A very good simple discussion of the subject may be found in Ernst Mach's *Mechanics* (translated by Thomas J. McCormack), Open Court Publishing Company, Chicago, 1907. Every serious student of mechanics should read this book of Mach's.

‡See Art. 53 for a discussion of active and inactive forces.

ample. in stretching a helical spring. *The simplest effect of a force, however, is the change which an unblanced force produces in the velocity of a body;* this effect is the simplest because it is independent of the nature of the body.* This effect is now universally adopted as the effect in terms of which force is fundamentally measured.

The study of the effects of unbalanced forces in modifying the motion of bodies constitutes the science of *dynamics*.

29. Types of motion and types of force action.† Motion, as it occurs in nature, is infinite in variety, but there are certain simple types of motion such as the forward motion of a boat or the rotatory motion of a wheel, and the discussion of these simple types of motion constitutes the science of mechanics. A body is said to perform *translatory motion* when every line in the moving body remains unchanged in direction. Thus a car moving along a straight track performs translatory motion. A body is said to perform *rotatory motion* when a certain line in the moving body remains fixed in position. This line is called the axis of rotation. Thus the flywheel of an engine performs rotatory motion.

A body may perform various types of motion simultaneously. It is better, however, to study each type of motion by itself, and some help is afforded towards the keeping of the types of motion clearly separated in one's mind by conceiving of ideal bodies as follows:

A *material particle* is an ideal body so small that the only sensible motion of which it is capable is translatory motion. The term material particle is used merely to direct one's attention to translatory motion, and any body whatever which performs translatory motion may be thought of as a particle if one wishes to think in such terms.

A *rigid body* is an ideal body which cannot alter its shape and

*This matter is discussed at some length on pages 6–11.

†Other types of motion and of force action are discussed later. Thus a variety of force actions are discussed in the chapter on elasticity, and the ideal *simple motion of flow* of a fluid is discussed in the chapter on hydraulics.

which is capable, therefore, of performing only translatory motion and rotatory motion. The term rigid body is used merely to exclude the idea of change of shape in the discussion of rotatory motion.

The use of ideal bodies in the development of mechanics may seem to be objectionable, but it is necessary to discuss one thing at a time and it is even more necessary to ignore the interminable array of minute effects which always accompany every physical phenomenon; an attempt to consider these minute details would complicate every engineering problem beyond the possibility of a practical solution. Thus for most practical purposes one may think of the motion of a railway car along a straight level track as simple translatory motion, whereas the actual motion involves the swaying and vibration of the car and the rattling of every loose part, it involves a complicated phenomenon of motion which is called journal friction, and it involves the yielding of the track and a whirling, eddying motion of the air, it is, in fact, infinitely complicated; but the railway engineer who, for example, is concerned with the design of a locomotive of adequate power sums up all of these effects in a rough estimate of the total frictional drag which the locomotive has to overcome.

Types of force action. To each type of motion there is a corresponding type of force action. Thus a force action which tends to produce translatory motion only is called a *linear force*, and that action of a force which tends to produce rotatory motion is called *turning force* or *torque.*

30. Center of mass. The simplest case of translatory motion is the motion of a body along a straight path, either with constant or varying velocity, as exemplified by the motion of a car along a straight track or the motion of a ship on a straight course. The most general case of translatory motion, however, is that in which a given point of the body describes any path whatever in any way whatever, but where every line in the body remains unchanged in direction as indicated in Fig. 25.

It is important to understand that there is a certain point in a body at which a single force must be applied to produce translatory motion without rotation; and it is important to understand that when a body does perform translatory motion without rotation, the body may be thought of as being concentrated at this point. This point is called *center of mass* of the body. Grasp a long slim stick at its middle and move it up and down and to and fro in any way but without changing the direction of the stick; it seems, with the eyes closed, as if the stick were a heavy body concentrated in the hand, or, in other words, the stick behaves as if it were all concentrated at its middle point which is its center of mass.*

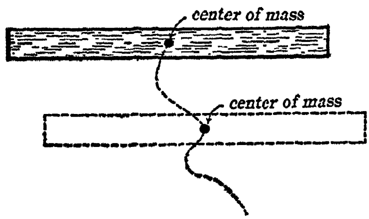


Fig. 25.

31. Displacement, velocity and acceleration.† When a particle moves from one position to another it is said to be *displaced*, and the distance (and direction) from the initial to the final position of the particle is called its *displacement*.

The displacement of a particle divided by the time during which the displacement takes place is called the *average velocity* of the particle during that time. A particle may move in any way whatever in making a given displacement and it is therefore evident that the *actual velocity* of the particle at successive instants during the displacement may be very different from the *average velocity*; but if the interval of time is extremely short (and, of course, the displacement small) then all irregularities

*Every student should perform this experiment for himself; no amount of argument will give him so vivid an idea of the meaning of the above statements concerning center of mass, namely (a) that a single force must be applied at the center of mass of a body to produce translatory motion and (b) that a body may be thought of as concentrated at its center of mass in so far as translatory motion only is concerned. Center of mass is discussed in Arts. 48, 49 and 50.

†The discussion of the time variation of velocity which is given in Art. 18 should be reviewed at this point.

vanish in accordance with the principle of continuity as stated in Art. 17. Therefore, the *average velocity* of a particle during a very short interval of time is the *actual velocity* of the particle at a given instant.

When the velocity of a particle is changing, the actual change during a given interval of time divided by the interval is called the *average acceleration* of the particle during the interval; and if the interval is very short, the acceleration so defined is the *actual acceleration* of the particle at a given instant.

The actual velocity of a particle at a given instant is of course never determined by an attempt to observe the displacement during a very short interval of time, and the actual acceleration of a particle at a given instant is of course never determined by an attempt to observe the change of velocity during a very short interval of time. The above definitions do not refer to *observation* but to *thinking*, as explained in Art. 17.

32. Newton's laws of motion. The laws of motion are discussed in detail on pages 2–11. A usual form of statement of these laws is as follows:

I. All bodies persevere in a state of rest, or in a state of uniform motion in a straight line, except in so far as they are made to change that state by the action of an unbalanced force.

II. (*a*) The acceleration of a particle is parallel and proportional to the unbalanced force acting on the particle.

(*b*) The acceleration which is produced by a given unbalanced force is inversely proportional to the mass of the particle.

III. Action is equal to reaction and in a contrary direction.

A clear understanding of the laws of motion is of utmost importance, and the following modified statements may serve to make their meaning clear.

(1) The first law describes the behavior of a particle upon which no unbalanced force acts. The behavior is simply that *the velocity of the particle does not change*, and, conversely, if a body move at uniform velocity in a straight line, the forces which act upon it are balanced.

(2) The second law describes the behavior of a particle when acted upon by an unbalanced force. The behavior is simply that *the particle gains velocity in the direction of the force at a rate which is (a) proportional to the force and (b) inversely proportional to the mass of the particle.*

When Newton made this statement he, of course, had it in mind that a force was measured by some effect other than acceleration, otherwise he could not have affirmed, except as a definition, that the acceleration which a force produces is proportional to the force. The production of acceleration is now adopted as the effect by which forces are measured.

The second law may be further paraphrased as follows: (*a*) The amount of velocity gained by a given particle in a given interval of time is proportional to the unbalanced force acting on the particle, and the gained velocity is parallel to the force. (*b*) The amount of velocity produced by a given unbalanced force in a given interval of time is inversely proportional to the mass of the particle upon which the force acts, and the gained velocity is, of course, parallel to the force.

A common form of statement of Newton's second law is that *the effect of a force is the same whether it acts alone or in conjunction with other forces*, meaning, of course, **the accelerating effect.*** To see that this statement of the second law is equivalent to the statements previously given, let it be understood that *any effect which is proportional to a cause may be divided into parts and each part assigned as the effect of a corresponding part of the cause.* Thus if the result of the labor of a number of men is proportional to the number of men, then it is justifiable from physical considerations to give each man an equal share of the profits; but if the result is *not* proportional to the number of men, it is *not* justifiable physically to give to each man an equal share of the profits. The principle of dividing cause and effect into parts which correspond

*Every schoolboy knows that, in general, the effect of a force is *not* the same whether it acts alone or in conjunction with other forces; two boys on thin ice may produce an effect which is absolutely different from anything which can be produced by the weight of one boy alone.

each to each *when cause and effect are proportional*, is called the **principle of superposition** and it runs through the whole science of physics and chemistry.

An example of Newton's second law is the familiar fact that two bricks fall at the same increasing velocity as one brick; the total force with which the earth pulls on the two bricks produces velocity at the same rate as the single force with which the earth pulls on one brick. That is, to double the mass of a body (two bricks instead of one) and at the same time to double the force acting on the body (pull of earth on two bricks instead of pull of earth on one brick) leaves the acceleration unchanged.

(3) The third law expresses the fact that a force is always due to the mutual action of two bodies, that this mutual action always consists of a pair of equal and opposite forces, and that one of these forces acts on body number one and the other upon body number two. The mutual force action between two bodies is called *action* in its effect upon the body which is being studied and *reaction* in its effect upon the body which is not being particularly studied, in the same way that a trade is called *buying* as it effects one person or *selling* as it affects the other person.

Inertia. That property of a particle by virtue of which it perseveres in a state of rest or in a state of uniform motion in a straight line when not acted upon by an unbalanced force is called *inertia*; and the word inertia is generally extended in its meaning to include, not only this passive property, but also the idea of *reluctance to gain velocity*. Thus a given unbalanced force would have to act for a longer time on a body of large mass than upon a body of small mass to produce a given velocity, that is, the body of large mass has the greater reluctance to gain velocity, or a greater inertia in the extended sense of that term.

33. Dynamic units of force. Formulation of the second law of motion. Having agreed to measure a force in terms of its effect in changing the velocity of a particle, we may choose as our unit of force that force which, acting as an unbalanced force

on unit mass, will produce unit velocity in unit time (unit acceleration). Thus the *dyne* is that force which, *acting for one second* as an unbalanced force on a one-gram particle, will produce a velocity of one centimeter per second (an acceleration of one centimeter per second per second); and the *poundal* is that force which, *acting for one second* as an unbalanced force on a one-pound particle, will produce a velocity of one foot per second (an acceleration of one foot per second per second). The dyne is the c. g. s. unit of force and it is much used; the poundal is seldom used.

Having adopted as our unit of force that force which will produce unit acceleration of unit mass, it is evident, from Newton's second law, that F units of force will produce F units of acceleration of a particle of unit mass, or F/m units of acceleration of m units of mass; that is, F/m is *equal* to the acceleration a which the force F produces or $F/m = a$, whence

$$F = ma \tag{1}$$

in which F is the value of an unbalanced force in dynes (or poundals), m is the mass of a particle in grams (or pounds), and a is the acceleration in centimeters per second per second (or feet per second per second).

Gravitational units of force. Weight. The force with which the earth pulls on a body is called the *weight* of the body. The precise meaning of the terms *mass* and *weight*, as these terms are used in scientific writing, is explained in Art. 5. The force with which the earth pulls on a one-pound mass (on a one-pound body) is called a *pound-of-force* or a *pound-pull*.* The pound-of-force is used as the unit of force in most practical work. Thus we speak of 5000 pounds-of-force, meaning a force equal to that with which the earth pulls on a 5000-pound body.

The pull of the earth on a one-pound body is about one-quarter of one per cent. greater at 60° north latitude than it is at 30° north latitude, and it is about one-sixteenth of one per cent. less at

*The *gram-of-force* or the *gram-pull* is defined in a similar manner.

15,000 feet above sea level than it is at sea level.* This slight variation in the value of the pound-of-force at different places on the earth is of no consequence, however, in those cases where the pound-of-force is used as a unit. Thus the tensile strength of a given grade of steel, repeatedly measured under conditions as nearly alike as it is possible to make them, will vary from, say, 100,000 pounds-of-force per square inch to 105,000 pounds-of-force per square inch, that is, the tensile strength of a given grade of steel is inherently indefinite (like the length of an angle-worm!), and a variation of a few tenths of one per cent. in the value of the unit of force is of no consequence whatever.

Relation between dynamic and gravitational units of force. A force of one dyne will produce an acceleration of one centimeter per second per second when it acts as an unbalanced force on a one-gram body; the pull of the earth on a one-gram body produces an acceleration of about 980 centimeters per second per second; therefore the pull of the earth on a one-gram body is equal to about 980 dynes.

A force of one poundal will produce an acceleration of one foot per second per second when it acts as an unbalanced force on a one-pound body; the pull of the earth on a one-pound body produces an acceleration of about 32.2 feet per second per second; therefore one pound-of-force is approximately equal to 32.2 poundals.

Let W be the force, expressed in dynamic units, with which the earth pulls on a body, then according to equation (1), we have

$$W = mg \tag{2}$$

in which W is the weight of the body in dynes (or poundals), m is the mass of the body in grams (or pounds), and g is the acceleration due to gravity expressed in centimeters per second per second (or in feet per second per second).

*The acceleration of gravity at 45° north latitude and at sea level is 980.6 centimeters per second per second, and the acceleration of gravity at latitude ϕ and at an elevation of H meters above sea level is:

$$g = 980.6\ (1 - 0.0026 \cos 2\phi - 0.0000002\, H)\ \text{cm./sec}^2.$$

Form of equation (1) when F is expressed in pounds-of-force, mass in pounds, and acceleration in feet per second per second. One pound-of-force will produce an acceleration of about 32.2 feet per second per second when it acts as an unbalanced force on a one-pound body, or an acceleration of $32.2/m$ feet per second per second when it acts on a body of which the mass is m pounds, and F pounds-of-force will produce F times as much acceleration, or an acceleration of $(32.2F)/m$ when it acts upon a body of which the mass is m pounds. That is to say, the acceleration a which is produced by F pounds-of-force acting upon a mass of m pounds is

$$a = \frac{32.2F}{m} \tag{3}$$

whence

$$F = \frac{1}{32.2} \cdot ma \tag{4}$$

or, using g for the acceleration of gravity in feet per second per second, we have

$$F = \frac{1}{g} \cdot ma \tag{5}$$

The slug as a unit of mass. Many engineers prefer to write equation (5) in the form

$$F = \left(\frac{m}{g}\right) a$$

and in order to bring this equation into the form of equation (1) these engineers speak of m/g as the mass of the body. *It must be kept in mind, however, that the mass of a body when so expressed is not expressed in pounds.* In buying sugar or coal by the pound the word pound is used in its legitimate sense as a unit of mass; and to express the mass of a body in terms of the ratio m/g is to adopt 32.2 pounds as the unit of mass. This unit of mass is sometimes called the "gee-pound" or the "slug." Throughout this text mass will be expressed in pounds, force in pounds-pull and acceleration in feet per second per second, except where c. g. s. units are used, and equation (5) will always be used when English units are employed.

34. Measurement of force. (*a*) *By the kinetic method.* The force (unbalanced) acting on a body may be calculated by equation (1), the mass of the body being known and the acceleration being determined by observation. This method for measuring

force cannot be realized in its simplicity, but it forms the basis of many physical measurements.

(*b*) *By the counter-poise method.* The strengths of materials are nearly always determined by applying, as the breaking force, the weight of a body or bodies of known mass, multiplied in a known ratio by a system of levers. The machine for carrying out such a test is called a *testing machine* and it is similar in many respects to the ordinary platform balance-scale.

(*c*) *By means of the spring scale.* The spring scale is an arrangement in which an applied force stretches a spring and moves a pointer over a divided scale. The movement of the pointer is proportional to the force, and, the movement for a known force being observed, the scale can be divided so as to

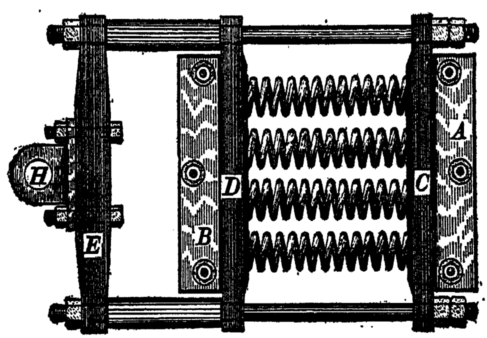


Fig. 26.

read the value of any force directly. The use of the spring-scale is exemplified in the measurement of the draw-bar pull of a locomotive. Figure 26 shows a scale designed for this purpose. The blocks A and B are rigidly fixed to the "dynamometer car" and the link H couples with the locomotive. A pull on H moves the cross bar C and compresses the springs, and a push on H moves the cross bar D and compresses the springs. The relative motion of E and B actuates a pointer which plays over a divided scale.

UNIFORMLY ACCELERATED TRANSLATORY MOTION.

35. Falling bodies. When a constant* unbalanced force acts upon a particle, the particle gains velocity at a constant rate. Such a particle is said to perform *uniformly accelerated motion.* A body falling freely under the action of the constant pull of the earth is, in so far as the friction of the air is negligible, an example of uniformly accelerated motion.

All bodies when falling freely gain velocity at the same rate, air friction being negligible. Thus two bricks together fall at exactly the same increasing speed as one brick alone. The doubled pull of the earth on the two bricks produces the same acceleration as the single pull of the earth on one brick. Doubling the force and doubling the mass leaves the acceleration unaltered.

Consider a particle which gains velocity at a constant rate of g centimeters per second per second, a falling body for example. The velocity gained in t seconds is

$$v = gt \tag{i}$$

Let v_1 be the initial velocity of the particle. Then $v_1 + gt$ is its velocity after t seconds, and its average† velocity during the t seconds is $\frac{1}{2}[v_1 + (v_1 + gt)]$ or $v_1 + \frac{1}{2}gt$; and the distance d

*Constant in magnitude and unchanging in direction.

†Let the constantly increasing velocity of a falling body be represented by the ordinates of a curve of which the abscissas represent elapsed times. The "curve".

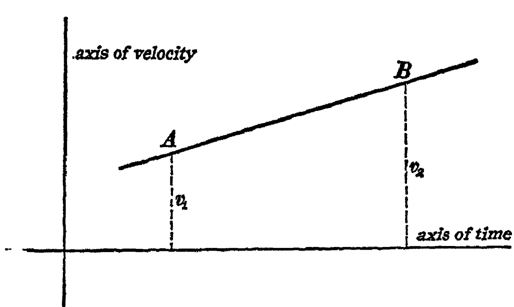


Fig. 27.

so plotted will be a straight line AB, Fig. 27, and the average ordinate of any portion AB of this line is equal to $\frac{1}{2}(v_1+v_2)$.

fallen by the particle during the t seconds is equal to the product of the average velocity by the time t. That is,

$$d = v_1t + \tfrac{1}{2}gt^2 \qquad \text{(ii)}$$

If v_1 is zero, equation (ii) becomes

$$d = \tfrac{1}{2}gt^2 \qquad \text{(iii)}$$

Eliminating t between equations (i) and (iii), we have

$$v = \sqrt{2gd} \qquad \text{(iv)}$$

which expresses the velocity of a body after it has fallen a distance d (initial velocity zero).

This discussion of falling bodies exemplifies the method of integral calculus. See footnote on page 38.

36. Projectiles. When the initial velocity v_1 of a body is zero or when it is vertical, we have the ordinary case of a falling body, and equation (ii) of Art. 35 can be solved by simple arithmetic, the only complication being that v_1 is to be considered negative when it is upwards. When the initial velocity v_1 is not vertical, as in the case of a tossed ball, the falling body is called a projectile. In this case the entire argument of Art. 35 holds

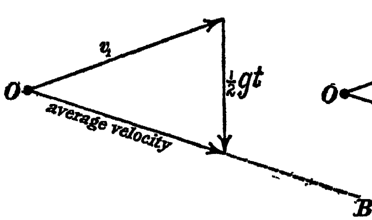


Fig. 28*a*.

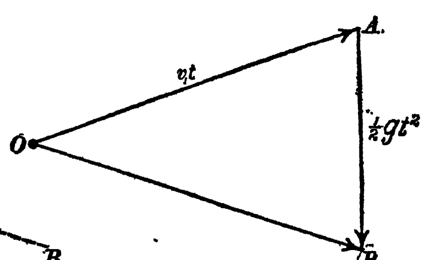


Fig. 28*b*.

good but geometric addition must be substituted for arithmetic addition. Thus the average velocity of a projectile during t seconds is equal to the *geometric sum*, $v_1 + \tfrac{1}{2}gt$, as shown in Fig. 28*a*, and after t seconds the projectile is on the line OB at a distance from O equal to t times the numerical value of the average

velocity. Or, one may find the position of the ball after t seconds on the basis of equation (ii), considering that v_1t is a distance in the direction of v_1, that $\frac{1}{2}gt^2$ is a distance vertically downwards, and that the sum $v_1t + \frac{1}{2}gt^2$ is a geometric sum as shown in Fig. 28*b*.

The orbit of a projectile is a parabola. This may be shown by choosing the x-axis of reference parallel to v_1 and the y-axis vertically downwards. Then $x = v_1t$ and $y = \frac{1}{2}gt^2$, whence, by eliminating t we have the equation of the parabola.

The hodograph to the orbit of a projectile is a vertical straight line. Draw the line OP, Fig. 29, representing the velocity of a projectile at a given distant, then, after t seconds, the vertical velocity gt will be gained, and the total velocity will be represented by OP'. Therefore, if we imagine the line OP to change so as to become OP' after t seconds and thus represent the changing velocity at each instant, then the end P will move vertically downwards at a constant velocity.

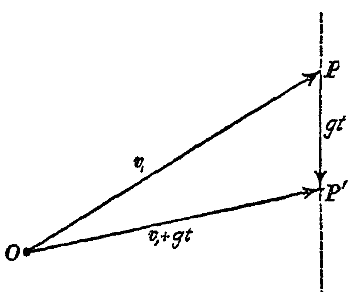


Fig. 29.

Range of a projectile. The horizontal distance reached by a projectile when it comes to the level of the gun on its downward flight is called the *range* of the projectile. The range of a projectile, ignoring the effects of air friction, is given by the equation

$$l = \frac{2v_1^2 \sin\theta \cos\theta}{g} \tag{i}$$

in which v_1 is the initial velocity of the projectile, θ is the angle between the direction of v_1 and a horizontal line, and g is the acceleration of gravity. This expression for l may be easily derived with the help of the relations shown in Fig. 30, namely,

$$l = v_1t \cos\theta$$

and

$$\tfrac{1}{2}gt^2 = v_1t \sin\theta$$

whence, eliminating t, we have equation (i).

37. Effect of air resistance on the motion of a projectile. Bodies which are projected through the air do not have a constant downward acceleration, because of the resistance which the air offers to their motion, and therefore the simple theory of projectiles above outlined is not applicable in practice. The limitations of this simple theory may be stated in a general way as follows:

(*a*) In the first place the above simple theory is *not* limited to the motion of an ideal particle. The pull of the earth upon a projectile *tends only to produce translatory motion and the center*

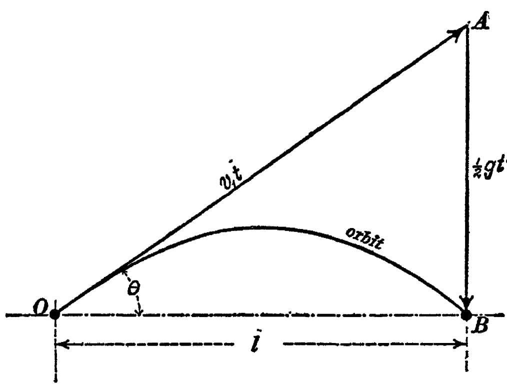


Fig. 30.

of mass of the body describes in every case a smooth parabolic curve in accordance with the discussion of Art. 36, air friction being ignored. Thus, if an iron bar is thrown through the air, the center of mass of the bar describes a smooth parabolic orbit; or if the bar is projected by hitting it a sharp blow with a hammer, the center of mass of the bar describes a smooth parabolic orbit. This illustrates a very important extension of the idea of a material particle, namely, we may call *any body* a material particle, *whatever the character of its motion may be*, the idea being to direct one's attention solely to that part of the motion of the body which is translatory.

(*b*) In the case of a heavy body moving slowly, for example, an iron ball tossed from the hand, the resisting force of the air is

very small compared with the weight of the body, and the motion of the body approximates very closely indeed to the ideal motion discussed in Art. 36.

(*c*) In the case of a light body, or in the case of a heavy body projected at high velocity, the resisting force of the air may be

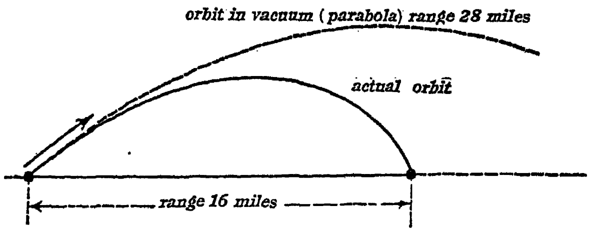


Fig. 31.

very large, so that the motion of such a body differs widely from the ideal motion described in Art. 36. Thus, Fig. 31 shows the actual orbit of the heavy projectile from a modern high power gun, and the dotted line shows what the orbit would be in a vacuum.

(*d*) The air friction on a rotating projectile generally gives rise to a force which pushes the projectile sidewise. This side force

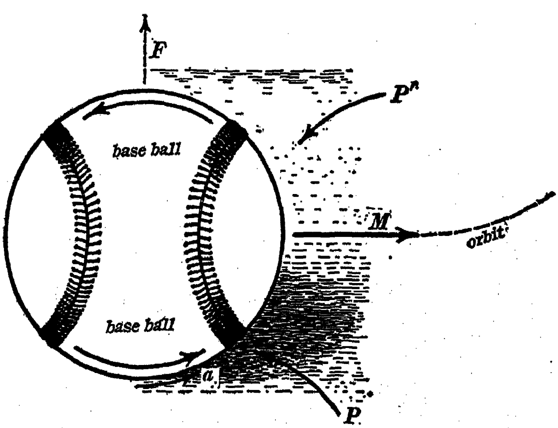


Fig. 32.

is the cause of the curiously curved orbit of a "split-shot" tennis ball, and of a base ball pitched by an expert pitcher. The

curved arrows, Fig. 32, show the direction of rotation of a base ball, the arrow M shows the direction of its translatory motion, the arrow F shows the side force above mentioned, and the dotted curve shows the curved orbit.*

TRANSLATORY MOTION IN A CIRCLE.

38. Velocity and acceleration of a particle moving steadily in a circular orbit. Consider a particle which makes, steadily, n revolutions per second in a circular orbit of radius r. The circumference of the orbit is $2\pi r$, and, inasmuch as the particle traverses the circumference n times per second, its velocity v is

$$v = 2\pi r n \tag{6}$$

The magnitude, or numerical value, of the velocity v is constant; but its direction is changing continuously, this continual change of direction of v involves acceleration, and the state of affairs at each instant during the steady motion of a particle in a circular orbit is most clearly shown by the use of the idea of the hodograph as explained in Art. 18. It is instructive, however, to discuss the motion of a particle in a circular orbit without explicit reference to the hodograph, as follows:

To determine the acceleration of a particle which is moving steadily in a circular orbit, it is necessary to consider the change of velocity during a very short interval of time. The circle, Fig. 33, represents the orbit of the particle, and at a given instant the particle is at P. At this instant the velocity v_1 of the particle is at right angles to PO and it is represented by the line $O'P'$ which is drawn from the fixed point O'. After the small lapse of time Δt, the particle will have moved a distance $v \cdot \Delta t$ to the point Q, and its velocity will be v_2, which is represented by the line $O'Q'$. The change of velocity Δv is evidently parallel to PO (or to QO, for it must be remembered that the time interval Δt is infinitely small), and, since the triangles OPQ and $O'P'Q'$ are similar, we have

*This matter is discussed again in the chapter on hydraulics.

$$\frac{\Delta v}{v} = \frac{v \cdot \Delta t}{r} \tag{i}$$

in which v is written for the common numerical value of v_1 and v_2, and $v \cdot \Delta t$ is the length of the infinitesimal arc PQ which is

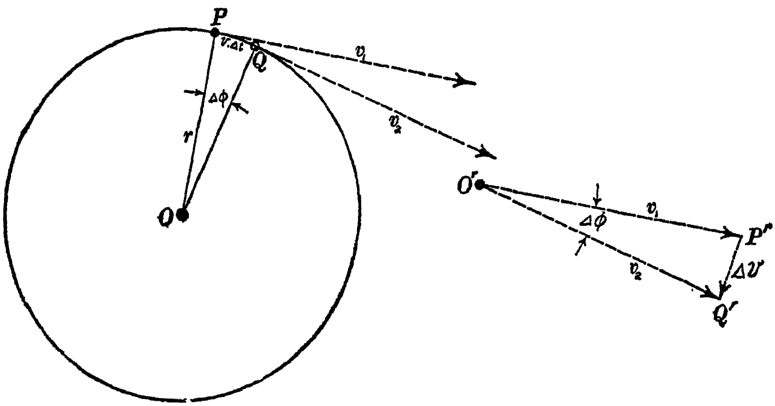


Fig. 33.

traversed by the particle during the time interval Δt. From equation (i) we have

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r} \tag{ii}$$

but, the change of velocity Δv divided by the time interval Δt during which the change takes place is the acceleration, so that, writing a for $\Delta v/\Delta t$, equation (ii) becomes

$$a = \frac{v^2}{r} \tag{7}$$

The direction of a is, of course, parallel to Δv, and Δv is parallel to PO. Therefore *a particle which moves steadily in a circular orbit of radius r has a steady acceleration towards the center of the circle, and this acceleration is equal to* v^2/r, *where v is the steady velocity of the particle.*

It is sometimes convenient to have a expressed in terms of r and n, thus we may substitute the value of v from equation (6) in equation (7) and we have

$$a = 4\pi^2 n^2 r \tag{8}$$

Force required to constrain a particle to a circular orbit. When a piece of metal is tied to a string and twirled in a circular orbit the string pulls steadily on the piece of metal, this pull of the string is an unbalanced force since no other force* acts on the piece of metal, and the value of the force in dynamic units is equal to the product of the mass of the particle and its acceleration, according to equation (1). Therefore we may substitute the value of a from equation (7) or equation (8) in equation (1) giving

$$F = \frac{mv^2}{r}\dagger \tag{9}$$

and

$$F = 4\pi^2 n^2 rm\dagger \tag{10}$$

where F is the force in dynamic units required to constrain a particle of mass m to a circular orbit of radius r, v is the velocity of the particle, and n is the number of revolutions per second.

39. Examples of motion in a circle. (*a*) A one-pound piece of metal twirled five revolutions per second in a circle four feet in radius would, according to equation (10), require a force of 3,948 poundals or about 123 pounds-weight to constrain it to its orbit.

(*b*) Each particle of a rotating wheel must be acted upon by an unbalanced force to constrain the particle to its circular path. If we consider only the rim of the wheel, neglecting the effect of the spokes, it is evident that the necessity of the unbalanced radial forces gives rise to a state of tension in the rim. The tension in a barrel hoop presses each portion of the hoop radially against the barrel staves, and the outward push of the staves balances

*Resistance of the air and force of gravity are here ignored.

†These equations express F in dynamic units, dynes or poundals as the case may be. If F is to be expressed in pounds-weight these equations become $F = \frac{1}{32.2} mv^2/r$ and $F = \frac{1}{32.2}(4\pi^2 n^2 rm)$, where m is the mass in pounds of the moving particle, v is its velocity in feet per second, r is the radius of the circle in feet, and n is the number of revolutions per second.

the radial force due to the tension of the hoop; but the tension in the rim of a rotating wheel produces an *unbalanced* radial force on each particle of the rim, and this force produces the radial acceleration of each part of the rotating hoop.

(*c*) The tension of a belt produces a radial force which presses the belt radially against the face of the pulley. When the belt and pulley are in motion, however, a portion of the belt tension produces the radial forces required to constrain the particles of the belt to their circular paths; the portion of the belt tension so used is proportional to the square of the velocity of the belt and inversely proportional to the radius of the pulley ($a = v^2/r$). Therefore, belts running at high speeds on small pulleys have a troublesome tendency to slip, unless the tension is very great.

(*d*) The centrifugal drier which is used in laundries and in sugar refineries is a rotating bowl AB, Fig. 34, with perforated

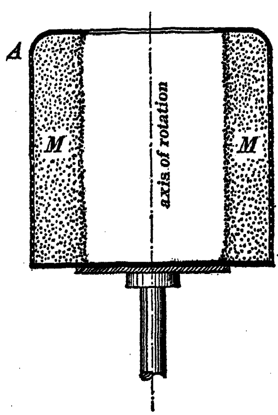


Fig. 34.

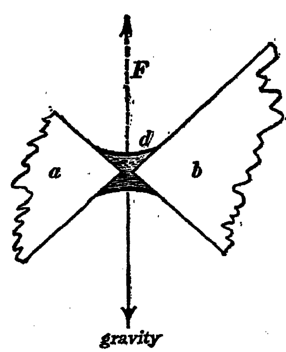


Fig. 35.

sides, in which the material MM to be dried is placed. The action of the centrifugal drier may be clearly understood as follows: Consider two solid particles a and b, Fig. 35, with a drop of water d clinging to them. Gravity, of course, pulls on the drop and the drop adheres to a and b so that the particles are able to exert on the drop a force F sufficient to balance gravity. In the centrifugal drier, however, the particles would have to exert upon

the drop a force equal to $4\pi^2n^2rm$, where r is the radius of the circular path described by the particles a and b, m is the mass of the drop, and n is the speed of the drier bowl in revolutions per second, and this force $4\pi^2n^2rm$ may be, say, 1000 times as great as the weight of the drop; but the drop does not have sufficient adherence to the particles to enable the particles to hold to it with so great a force, and the result is that the drop is *not* constrained to the circular path, but flies off tangentially. The action of the centrifugal drier is as if a piece of wet cloth were jerked so quickly to one side as to leave the water behind.

(*e*) A locomotive on a railway curve describes a circular path and an unbalanced horizontal force (equal to mv^2/r) must push the locomotive towards the center of the curve in order that the

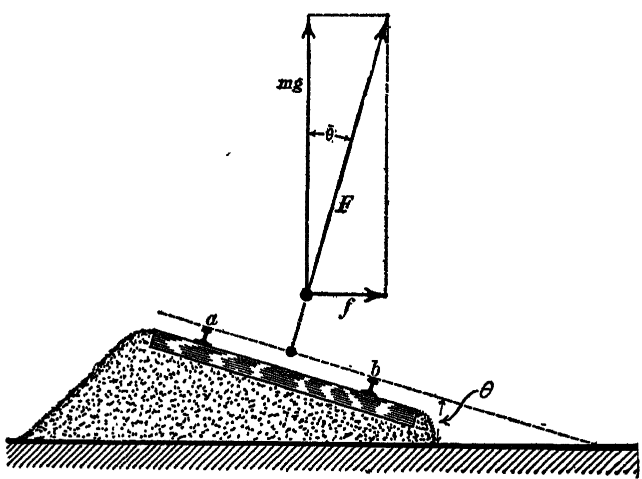


Fig. 36.

locomotive may follow the curve, and, of course, this horizontal force must be exerted on the locomotive by the track. It is desirable, however, that the total force with which the track pushes on the locomotive (which is equal and opposite to the force with which the locomotive pushes on the track) shall be perpendicular to the plane of the track, and, therefore, the outside rail is always raised on a railway curve.

Let us consider the proper elevation to be given to the outside rail when the velocity v of the locomotive and the radius r of the curve are given. The rails are shown at a and b in Fig. 36, F is the total force that must act upon the locomotive, and the angle θ is the required elevation.*

The vertical component of F is what sustains the locomotive against gravity. Therefore this vertical component is equal to mg where m is the mass of the locomotive and g is the acceleration of gravity. That is:

$$F \cos \theta = mg \tag{i}$$

The horizontal component of F is the unbalanced force which constrains the locomotive to its circular path. Therefore this horizontal component is equal to v^2m/r according to equation (9). That is:

$$F \sin \theta = \frac{v^2m}{r} \tag{ii}$$

Therefore, dividing equation (ii) by equation (i), member by member, we have

$$\tan \theta = \frac{v^2}{rg} \tag{iii}$$

When a locomotive is traveling on a curve it is evident that the whole locomotive is *rotating* about a vertical axis at such an angular speed that if the curve were a complete circle the locomotive would make one rotation about a vertical axis every time it traversed the circular curve. Therefore a locomotive traveling on a curve does not perform pure translatory motion; but here again is an instance where the translatory motion may be

*Some of the members of every class in elementary mechanics seem to have the idea that the tendency is for the outer wheels of a carriage to rise off the ground when the carriage is driven rapidly around a curve. This idea comes from the fact that the outer wheels of a carriage must be elevated (by raising the outer part of the road) to make the carriage ride around the curve with perfect safety, and a bicycle rider leans inwards in order that he may ride around a curve without falling over. The leaning over of a bicycle rider as he rounds a curve is not due to the mechanical actions involved but to the deliberate control of the rider.

considered by itself, for as long as the locomotive is *on* the curve, its rotating motion is *constant* and introduces no complication.

When a locomotive suddenly enters a curve from a straight portion of track, the rotatory motion of the locomotive about a vertical axis must be suddenly established, and in order to establish this rotatory motion an excessively great horizontal side-force must act on the front wheels of the locomotive; or in other

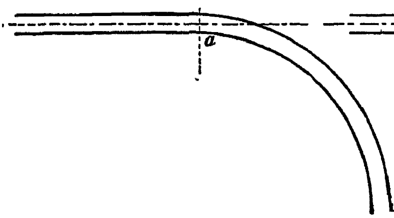


Fig. 37.

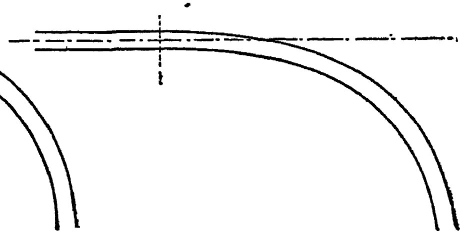


Fig. 38.

words, the front wheels of a locomotive must push with excessive force against the outer rail when the locomotive enters a curve suddenly. This action is called *nosing*, and it is especially troublesome in the case of a locomotive having a short wheel base. When the wheel-base of a locomotive is long, a much smaller side force need act upon the front wheels of the locomotive when it enters a curve, and therefore the nosing is less troublesome. The electric locomotives of the New York, New Haven & Hartford Railroad when first constructed had very short wheel-bases, and considerable trouble was encountered until pilot trucks were placed at both ends.

Fig. 37 shows a straight portion of a track changing abruptly to a circular curve at the point a, and Fig. 38 shows the same straight portion changing gradually into the same circular curve. The slow transition from straight to curved track in Fig. 38 constitutes what is called an *easement curve*, the object of which is to avoid the effects of abrupt entry of locomotive into a curved portion of track, as above described.

40. The rotating hoop. It is pointed out in the above examples of circular motion that the radial forces which constrain

the particles of the rim of a rotating wheel to their circular paths are (ignoring effect of spokes) due to a state of tension in the rim. The tension of the rim is the force T with which any portion of the rim pulls on a contiguous portion. Let the circles, Fig. 39, represent a hoop of a radius r rotating n revolutions per second about the axis C, let the *mass per unit length* of the rim of the hoop be m', and let F be the unbalanced force pulling radially inwards *on each unit length* of the rim (due to the tension in the rim). Consider a very short portion of the rim of length $r \cdot \Delta\varphi$, which subtends the angle $\Delta\varphi$ as shown. The

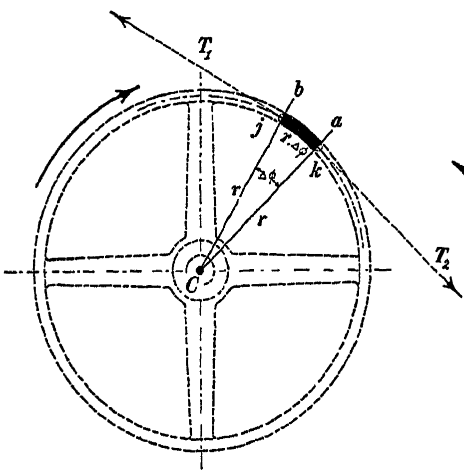


Fig. 39.

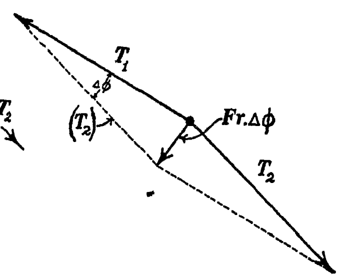


Fig. 40.

unbalanced force pulling the portion $r \cdot \Delta\varphi$ radially inwards is $F \times r \cdot \Delta\varphi$. Let T_2 be the force with which portion k pulls on the given portion $r \cdot \Delta\varphi$ at the point a, and let T_1 be the force with which the portion j pulls on the given portion $r \cdot \Delta\varphi$ at the point b. The forces T_1 and T_2 are equal, numerically, and they are tangent to the circle at a and b respectively. Draw the two lines T_1 and T_2, Fig. 40, parallel to T_1 and T_2, Fig. 39, and complete the parallelogram of which T_1 and T_2 are the sides. The diagonal of this parallelogram represents the resultant of T_1 and T_2, this resultant is the total unbalanced force $(F \times r \cdot \Delta\varphi)$

which acts on the given portion $r \cdot \Delta\varphi$ of the rim, and from the similar triangles of Fig. 39 and Fig. 40 we have

$$\frac{r \cdot \Delta\varphi}{r} = \frac{Fr \cdot \Delta\varphi}{T}$$

in which T is written for the common numerical value of T_1 and T_2. Therefore

$$F = \frac{T}{r} \qquad \text{(i)}$$

That is, the unbalanced inward pull on each unit length of a hoop is equal to the tension of the hoop divided by its radius.

Now the mass of unit length of the rim is equal to m'; and, according to equation (10), the force which must be pulling radially inwards on unit length of the rim to constrain it to its circular path, is equal to $4\pi^2n^2r$ times its mass m'. Therefore writing $4\pi^2n^2rm'$ for F in equation (i) we have

$$T = 4\pi^2n^2r^2m' \qquad (11)$$

in which T is the tension in dynes in a rim r centimeters in radius, rotating n revolutions per second, and m' is the mass in grams of one centimeter of the rim. If m' is expressed in pounds mass per foot of rim, r in feet, n in revolutions per second and T in pounds-weight, then

$$T = \frac{1}{32.2}(4\pi^2n^2r^2m')$$

Example. The rim of a large flywheel has a mass of 250 pounds per foot, the radius of the wheel is 15 feet, the wheel rotates one revolution per second, and the tension of the rim (neglecting the effect of the spokes) is 69,350 pounds-weight.

TRANSLATORY HARMONIC MOTION.

41. Definition of harmonic motion. Utility of the idea. Simple harmonic motion is the projection on a fixed straight line of uniform motion in a circle. Consider a point P', Fig. 41, moving uniformly around a circle of radius r at a speed of n

revolutions per second, *the point P, which is the projection of P' on the line CD, performs simple harmonic motion.*

Vibration or cycle. One complete up-and-down movement of the point P, Fig. 41, is called a *vibration* or a *cycle.*

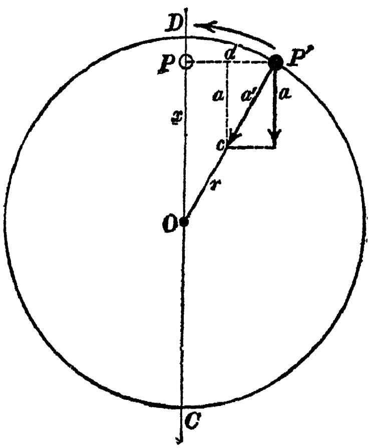


Fig. 41.

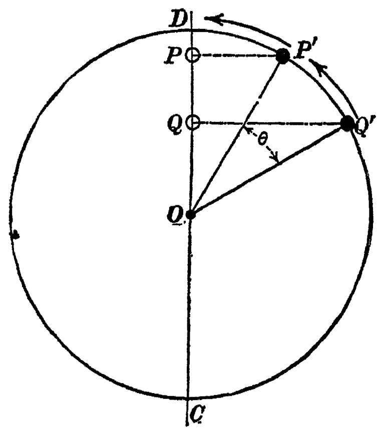


Fig. 42.

Frequency. The number of vibrations, or cycles, per second is called the *frequency* of the oscillations of the point P; this is, of course, equal to the number of revolutions per second of the point P'.

Period. The time required for the particle to complete one whole vibration, or cycle, is called the *period* of the harmonic motion. The relation between the frequency n and the period τ is obviously

$$n = \frac{1}{\tau} \tag{12}$$

Equilibrium position. When the vibrating particle P, Fig. 41, is at the point O, no force acts upon it, as explained below; the point O is therefore called the *equilibrium position* of the vibrating particle.

Amplitude. The maximum distance from O reached by the vibrating particle is called the amplitude of its oscillations. This amplitude is equal to the radius r of the circle in Fig. 41.

Phase difference. Consider two points P' and Q', Fig. 42, both making n revolutions per second around the circle so that the angle θ is constant. The two oscillating particles P and Q are then said to *differ in phase* and the angle θ is called their *phase difference.*

The ideas involved in the peculiar type of motion which is performed by the particle P, in Fig. 41, are used throughout the study of oscillatory motion and wave motion. Thus the prongs of a vibrating tuning fork perform simple harmonic motion; the motion of a pendulum bob is, approximately, simple harmonic motion; when a rod, or a beam, or a bridge oscillates in the simplest possible manner, each particle of the rod, or beam, or bridge performs simple harmonic motion; when wave-motion of the simplest kind spreads through a body each particle of the body performs simple harmonic motion.

An example of simple harmonic motion in which all of the details in Fig. 41 are reproduced, is the motion of the cross-head of a steam engine with a long connecting rod. The crank pin moves at sensibly uniform speed in a circle, one component, only, of this motion is transmitted to the cross-head by the long connecting rod, and the cross-head moves to and fro in the manner of the point P in Fig. 41.

42. Acceleration of a particle in harmonic motion. The velocity of the point P in Fig. 41 is the vertical component of the velocity of the point P', and the acceleration a of the point P is the vertical component of the acceleration a', therefore, from the similar triangles $P'OP$ and $P'cd$ of Fig. 41 we have

$$\frac{a}{a'} = \frac{x}{r}$$

where x is the distance OP, and, since $a' = 4\pi^2n^2r$, according to equation (8), we have

$$a = -4\pi^2n^2x. \tag{13}$$

The minus sign is introduced for the reason that a is downwards

(negative) when x is upwards (positive), and this sign has nothing to do with the numerical relations under discussion.

The force which must act on the particle P, Fig. 41, to cause it to move in the prescribed manner, is at each instant equal to ma, according to equation (1), therefore

$$F = -4\pi^2 n^2 mx \tag{14}$$

in which m is the mass of the oscillating particle, x is the distance of the particle from its equilibrium position at a given instant, n is the frequency of the oscillations, and F is the force which must act on the particle at the given instant.

The quantities n and m in equation (14) are constant. Therefore equation (14) indicates that *the force F which must act at each instant on a particle in harmonic motion is proportional to the distance x of the particle from its equilibrium position*, that is, we may write

$$F = -kx \tag{15}$$

where

$$k = 4\pi^2 n^2 m \tag{16}$$

or using $1/\tau$ for n, where τ is the period of one complete oscillation, we have

$$k = \frac{4\pi^2 m}{\tau^2} \tag{17}$$

43. Examples of the application of equations (15), (16) and (17). (*a*) *Application to a weight attached to the end of a flat spring.* A weight of mass m is fixed to one end of a flat steel spring S, the other end of which is clamped in a vise as shown in Fig. 43. If the weight M is pushed to one side through a distance x, the spring exerts a force F which urges the weight back towards its equilibrium position *and this force is proportional to x*. Therefore, we may write

$$F = -kx \tag{i}$$

in which k is a constant, the value of which may be determined by observing the force required to *hold* the weight at a measured distance x from its equilibrium position.

Now since equation (i) is identical to equation (15) it is evident that the weight, once started, will perform simple harmonic motion.

(*b*) *Application to the simple pendulum.* The simple pendulum is an ideal pendulum consisting of a particle P, Fig. 44, suspended

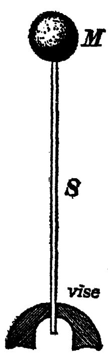


Fig. 43.

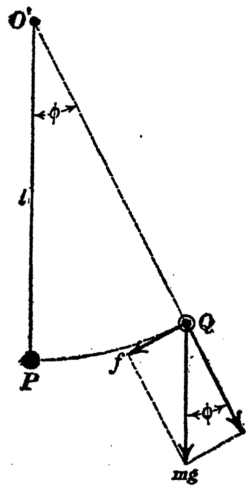


Fig. 44.

from a fixed point O by a string l of which the mass is negligible. If the particle P is moved to one side and released it will oscillate back and forth. *It is desired to show that these oscillations are simple harmonic oscillations, and that the period of one complete oscillation is equal to* $2\pi\sqrt{l/g}$, *where g is the acceleration of gravity.*

Let Q be the position of the oscillating particle at a given instant. The length x of the circular arc PQ is equal to $l\varphi$, and the component Qf of the force mg with which gravity pulls downwards on the particle, is equal to $mg \sin \theta$ or, *if the angle* φ *is very small*, this force is equal to $mg \cdot \varphi$, or to mg/l times φl, or to mg/l times x. Therefore, remembering that the force $Qf(= F)$ is to the left when the arc $PQ(= x)$ is to the right, we have

$$F = -\frac{mg}{l} \cdot x.$$

But this equation is identical in form to equation (15) since mg/l is constant, therefore the pendulum bob in Fig. 44 performs simple harmonic motion, and the equation expressing the period of the oscillations may be found by substituting mg/l for k in equation (17). In this way we find

$$\frac{mg}{l} = \frac{4\pi^2 m}{\tau^2}$$

or

$$\tau = 2\pi\sqrt{\frac{l}{g}} \tag{18}$$

44. Harmonic motion represented by a curve of sines. If the point P', Fig. 45, moves around the circle at uniform speed, the angle $P'OA$ is proportional to elapsed time, and it may be written ωt where ω is a constant and t is elapsed time reckoned from the instant that P' was at A. Therefore we may write

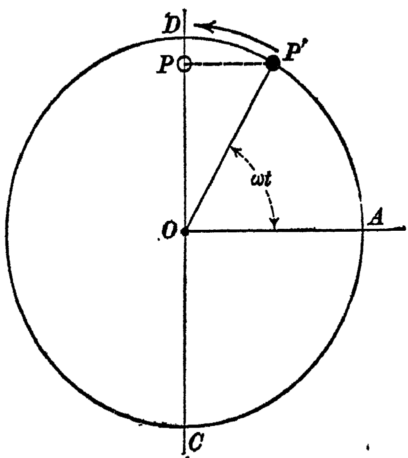


Fig. 45.

$$x = r \sin \omega t. \tag{19}$$

That is, the distance of the vibrating particle P from its equilibrium position O is proportional to the sine of a uniformly increasing angle, and if values of x be plotted as ordinates and the corresponding values of t (or ωt) as abscissas, we will have a curve of sines as shown in Fig. 46. If a fine pointer be attached to the prong of a tuning fork, the pointer may be made to trace a curve of sines by setting the fork in vibration and drawing the pointer uniformly across a piece of smoked glass.

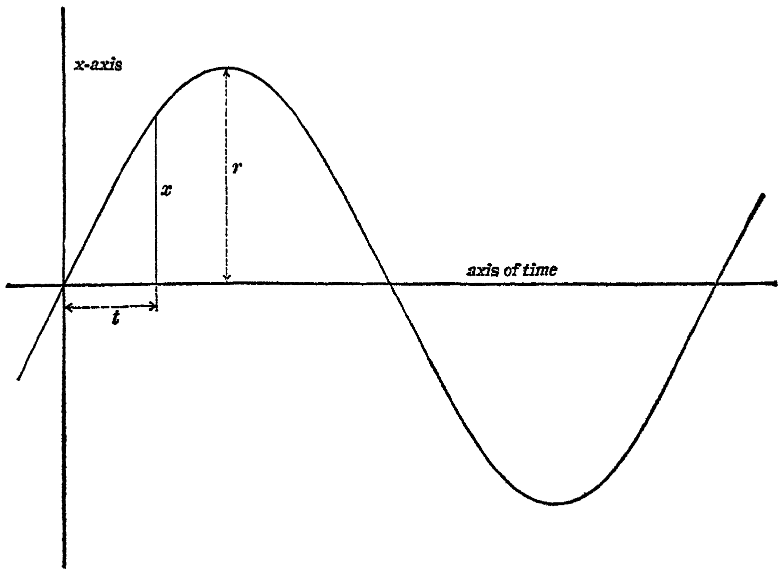


Fig. 46.

SYSTEMS OF PARTICLES.

45. System of particles. The ideas of translatory motion may *conceivably* be extended so as to serve as a basis for the description of any motion of any body or substance, by looking upon the body or substance as a collection of particles and considering the varying position, velocity, and acceleration of each particle. A collection of particles treated in this way is called a *system of particles* or simply a *system.* Thus a rotating wheel is a system of particles, a portion of flowing water is a system of particles, a given amount of a gas is a system of particles. The word *configuration* is used when we wish to refer to the relative positions of the particles of a system; thus the configuration of a system is said to change when the particles change their relative positions,

A *closed system* is a system no particle of which has any force acting on it from outside the system. There is no such thing in nature as a closed system, but the conception is useful nevertheless.

The cases in which it is not only conceivably possible, but

actually feasible to study more or less complicated types of motion by treating the moving substance as a system of particles, are as follows:

(*a*) The case in which the system consists of very few bodies and where each body may be treated as a particle.* This case is exemplified by the sun and planets.

(*b*) The case in which the particles of a system move in a perfectly regular or orderly way. Thus the particles of a rotating wheel move in an orderly fashion, the particles in a smoothly flowing liquid move in an orderly fashion, the particles of a vibrating string move in an orderly fashion, the connected parts of any machine such as a steam engine or a printing press move in an orderly fashion. Any system in which orderly motion takes place is called a *connected system.*

(*c*) The case in which the particles of a system move in utter disorder, without any connection whatever with each other. In this case it would evidently be impossible to consider the actual motion of each particle, in fact the only possible treatment of such a system is a treatment based on the idea of *averages* and *probable departures therefrom.* Thus the very important kinetic theory of gases has been built up on the hypothesis that a gas consists of innumerable disconnected particles in disordered motion.

46. Momentum. In the discussion of a system, the product mv of the mass of a particle and its velocity is of sufficient importance to warrant its receiving a name; it is called the *momentum* of the particle, it is a vector and its direction is the same as the velocity v of the particle.

When an unbalanced force acts upon a particle, of course the momentum of the particle changes; *the rate of change of the momentum is equal and parallel to the force.* This is evident when we consider that a change of velocity Δv means a *change of momentum equal to* $m \cdot \Delta v$, which, divided by the elapsed time Δt, gives the rate of change of momentum; but $\Delta v / \Delta t$ is equal

*Or where each body is a *connected system,* see (*b*).

to acceleration, so that $m \, . \, \Delta v/\Delta t$ is equal to mass times acceleration, and this is equal to the unbalanced force, according to equation (1), Art. 33.

The mutual force-action of two particles cannot change the total momentum of the two particles. This is evident when we consider that the mutual force-action of two particles consists of two equal and opposite forces (action and reaction), so that while one particle gains momentum in one direction, the other particle gains momentum in the opposite direction at the same rate. The constancy of the total momentum of two particles, insofar as their mutual force-action is concerned, is called the *principle of the conservation of momentum.*

The principle of the conservation of momentum applies to any number of particles insofar as their mutual force-actions are concerned. The total momentum of the particles of a system is never changed by the mutual force-actions within the system, or, in other words, the total momentum of a closed system is constant.

47. Impact. Consider two particles of which the masses are m_1 and m_2, and the velocities v_1 and v_2, respectively. The combined momentum of the two particles is $m_1v_1+m_2v_2$. If the bodies collide, their velocities may change. Let V_1 and V_2 be the respective velocities after impact. Then $m_1V_1+m_2V_2$ is the total momentum of the bodies after impact, and by the principle of the conservation of momentum, we have

$$m_1v_1+m_2v_2=m_1V_1+m_2V_2 \quad \text{(a vector equation)} \tag{i}$$

Impact of inelastic balls.—When an inelastic ball, such as a ball of soft clay strikes squarely against another, the two balls move after impact as a single body so that V_1 and V_2 are equal, and this common velocity after impact is completely determined by equation (i).

Impact of perfectly elastic balls.—Consider two elastic balls moving at velocities v_1 and v_2 in the same straight line (v_1 and v_2 being opposite in sign if the balls are moving in opposite directions). Let the masses of the balls be m_1 and m_2 respectively.

When these balls collide they are distorted, and at a certain instant the distortion reaches a maximum, after which the balls rebound from each other and the distortion is relieved. *When the distortion of the two balls has reached its maximum, the two balls are at the instant moving at common velocity c, which is determined by the equation*

$$(m_1+m_2)c=m_1v_1+m_2v_2 \tag{ii}$$

During the time that the balls are being distorted, which time we shall call the first half of the impact, the first ball loses* an amount of velocity (v_1-c) and the second ball loses* an amount of velocity (v_2-c). During the time that the balls

*The velocity c lies between v_1 and v_2 so that if (v_1-c) is positive then (v_2-c) must be negative.

are being relieved from distortion, which time we shall call the second half of the impact, they are assumed to act on each other with precisely the same series of forces as during the first half of the impact, only in a reverse order. This is what is meant by the assumption that the two balls are perfectly elastic. Therefore during the second half of the impact, each ball loses the same amount of velocity as it lost during the first half of the impact, that is, the total loss of velocity by the first ball is $2(v_1-c)$ and the total loss of velocity by the second ball is $2(v_2-c)$, so that

$$V_1=v_1-2(v_1-c)$$

and

$$V_2=v_2-2(v_2-c)$$

or

$$V_1=2c-v_1 \tag{iii}$$

and

$$V_2=2c-v_2 \tag{iv}$$

in which V_1 and V_2 are the respective velocities of the balls after impact.

Substituting the value of c from equation (ii) in equations (iii) and (iv) we have

$$V_1=\frac{m_1v_1+2m_2v_2-m_2v_1}{m_1+m_2} \tag{v}$$

$$V_2=\frac{2m_1v_1+m_2v_2-m_1v_2}{m_1+m_2} \tag{vi}$$

The simplest case is where $m_1=m_2$ and where $v_2=0$, that is where the balls are similar, and where the first ball only is in motion before impact. In this case the result may be derived from equations (v) and (vi) but it is more instructive to derive the result anew. The common velocity c at the middle of the impact is equal to $\frac{1}{2}v_1$. That is, the first ball loses half its velocity and the second ball gains an equal amount of velocity during the first half of the impact. During the second half of the impact the first ball loses the remainder of its velocity and comes to a standstill, and the second ball gains once more an equal amount of velocity so that its velocity is now equal to the initial velocity v_1 of the first ball. That is, when an elastic ball *A* strikes squarely against a similar stationary ball *B*, the ball *A* stops, and the ball *B* moves on with the full original velocity of *A*. If *A* is heavier than *B*, then both balls move in the same direction after the impact. If *B* is heavier than *A*, then *A* moves backwards, or has a negative velocity after the impact.

48. Motion of the center of mass of a system. The center of mass of a system has been defined in physical terms in Art. 30. The center of mass of a body of uniform density is at the geometrical center of the body. The center of mass of two particles lies on the line joining them, and its distance from each particle is inversely proportional to the mass of the particle. Thus the center of mass of the earth and moon is on the line joining the center of the earth and the center of the moon, and it is about 80 times as far from the center of the moon as it is from the

center of the earth (3,000 miles from the center of the earth), inasmuch as the mass of the earth is about 80 times as great as the mass of the moon.

The center of mass of a system remains stationary, or continues to move with uniform velocity in a straight line, if the vector sum of all of the forces which act on the system is zero.

For example, consider an emery wheel mounted on a shaft and rotating at high speed. If the center of mass of the wheel lies in the axis of the shaft, it, of course, remains stationary as the wheel rotates, and the only force that need be exerted on the shaft by the bearings is the steady upward force required to balance the downward pull of the earth on the wheel. A rotating machine part is said to be balanced when its center of mass is in its axis of rotation.

When the center of mass of a system is not stationary, and does not move with uniform velocity in a straight line, then the vector sum F_s of the forces which act on the system is not zero.

In fact, the acceleration A of the center of mass of a system of particles, the vector sum F_s of the forces which act on the system, and the total mass M of the system are related to each other precisely in the same way as the acceleration, force, and mass of a single particle. That is, as fully explained in Art. 50, we have

$$F_s = MA \tag{20}$$

Example 1. Consider an emery wheel of which the center of mass lies at a distance r to one side of the axis of rotation, then, as the wheel rotates, the center of mass describes a circular path of radius r, the acceleration of the center of mass is equal to $4\pi^2n^2r$ at each instant, and a side force equal to $4\pi^2n^2rM$ and *parallel to r at each instant* must act on the axle to constrain the center of mass to its circular path, precisely as if the entire mass of the wheel were concentrated at its center of mass.

Example 2. The centrifugal drier consists of a rapidly rotating bowl mounted on top of a vertical spindle, and the materials to be dried are placed in this bowl. It is impossible to keep the

bowl and contents even approximately balanced, so that, if the spindle were carried in a rigid bearing, the machine would be disabled in a short time because of the very great forces that would be brought into play in constraining the center of mass of bowl and contents to move in a circular path. This difficulty is obviated by supporting the spindle at the lower end only, in a long bearing mounted on springs to hold it approximately vertical. The bowl, contents, and spindle then rotate about a line passing through their center of mass and through the center of the flexible bearing, and, although the bowl and spindle seem to wobble badly (inasmuch as they do not rotate about the axis of figure), nevertheless the machine runs quite smoothly, producing but little vibration in the supporting frame.

Example 3. If two balls, which are tied together with a short string, are thrown in such a way that the string is kept stretched while the balls revolve rapidly about one another, a certain point of the string will describe a smooth parabolic curve, just as a simple projectile would do. This point of the string is the center of mass of the two balls. The center of mass of the earth and the moon describes an elliptic orbit about the sun once a year, while the earth and moon rotate about their center of mass once every lunar month, in a manner very similar to the motion of the two balls just described.

49. The balancing of a rotating machine part. Any part of a machine which is to rotate rapidly must be adjusted so that its center of mass lies in the axis of rotation.* This adjustment is called *balancing*, and a machine part so adjusted is said to be balanced. A machine part which is to be balanced, a dynamo armature for example, is mounted on its shaft and the ends of the shaft are placed upon two straight level rails. If the center of mass is in the axis of the shaft, the whole will stand in equilibrium in any position; whereas, if the center of mass is not

*A machine part which is long in the direction of the shaft upon which it rotates may have its center of mass in the axis of the shaft and yet the bearings may have to exert constraining forces upon the shaft as the part rotates. A long cylinder loaded on opposite sides at the two ends is an example.

in the axis of the shaft, the whole will come to rest with the center of mass at the lowest possible position, and material is removed from one side or added to the other side until the center of mass is in the center of the shaft.

Figure 47 shows a wheel mounted on a pair of balancing rails. Such a pair of balancing rails is a prominent feature in a shop where the fly-wheels of large engines have to be balanced.

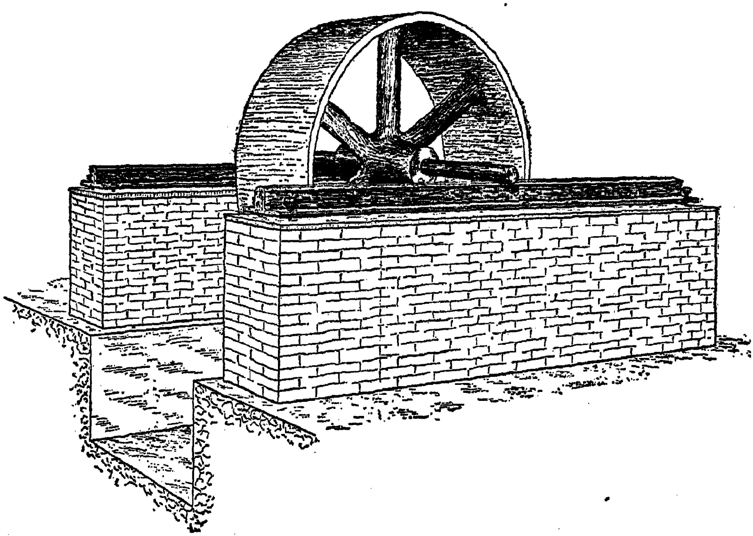


Fig. 47.

50. Equations of center of mass. The position of the center of mass of a system of particles may be expressed in terms of the positions and masses of all of the particles in the system as follows: Let x be the x-coordinate of a particle whose mass is m, let x' be the x-coordinate of a particle whose mass is m', let x'' be the x-coordinate of a particle whose mass is m'' and so on, then the sum $mx+m'x'+m''x''+$etc., divided by the total mass of the system, namely, $m+m'+m''+$etc., gives the x-coordinate of the center of mass of the system. That is, the x-coordinate of the center of mass is

$$X = \frac{\Sigma mx}{\Sigma m} \tag{21}$$

and exactly similar expressions may be formulated for the y-coordinate and for the z-coordinate of the center of mass.

If the origin of coordinates is at the center of mass of the system then, of course, X is equal to zero, and equation (21) becomes

$$\Sigma mx = 0 \tag{22}$$

In order to show that equation (20) is true, it is sufficient to consider only the x-component of A, and the x-components of the accelerations of the respective particles. The x-component of A is d^2X/dt^2 and the x-components of the accelerations of the respective particles are d^2x/dt^2, d^2x'/dt^2, d^2x''/dt^2 and so on. Therefore, writing M for Σm in equation (21), and differentiating twice with respect to time, we have

$$M\frac{d^2X}{dt^2} = m\frac{d^2x}{dt^2} + m'\frac{d^2x'}{dt^2} + \frac{d^2x''}{dt^2} + \text{etc.},$$

but $m(d^2x/dt^2)$ is the x-component of the force acting on the particle m, $m'(d^2x'/dt^2)$ is the x-component of the force acting on the particle m' and so on, so that the right-hand member of this equation is the sum of the x-components of all the forces acting on the particles of the system, and this is equal to the sum of x-components of all of the *external* forces acting on the particles of the system, inasmuch as mutual force-actions between the particles of the system cancel out of this sum because such mutual force-actions consist of pairs of equal and opposite forces. Therefore, the right hand member of the above equation is the x-component of the total external force F_s which acts on the system and the above equation reduces to

$$M \text{ times } x\text{-component of } A = x\text{-component of } F_s \qquad \text{(i)}$$

and we may show in exactly the same way that

$$M \text{ times } y\text{-component of } A = y\text{-component of } F_s \qquad \text{(ii)}$$

and

$$M \text{ times } z\text{-component of } A = z\text{-component of } F_s \qquad \text{(iii)}$$

These three equations are equivalent exactly to the single vector equation (20).

PROBLEMS.

52. A train having a mass of 350 tons (2,000 pounds) starting from rest reaches a speed of 50 miles per hour in 2½ minutes. What is the average pull of the locomotive during 2½ minutes, dragging forces of friction being neglected? Ans. 10,700 pounds-weight.

53. The above train moving at a speed of 50 miles per hour is brought to a standstill in 16 seconds by the brakes. What is the average retarding force in pounds-weight due to the brakes? Ans. 100,300 pounds-weight.

54. An elevator reaches full speed of 8 feet per second 2½ seconds after starting. With what average force in pounds-weight does a 160-pound man push down on the floor while the elevator is starting up? The elevator is stopped (when moving up at full speed) in 1½ seconds. With what average force in pounds-weight does a 160-pound man push down on the floor

while the elevator is stopping? Ans. 176 pounds-weight while the elevator is starting up; 133.3 pounds-weight while the elevator is stopping.

Note. In the first case the upward push of the floor on the man exceeds the weight of the man by the amount which is necessary to produce the upward acceleration; in the second case the weight of the man exceeds the upward push of the floor by the amount which is necessary to produce the downward acceleration.

The use of D'Alembert's principle simplifies the argument of this problem greatly. Consider the case in which the elevator is gaining velocity upwards. In this case an unbalanced upward force equal to $1/32.2 \cdot ma$ must be acting on the man. Therefore introducing a fictitious downward force equal to $1/32.2 \cdot ma$, and proceeding as in a problem in statics, we consider that the upward force exerted on the man by the platform must be equal to the total downward force acting on the man, namely, the weight of the man plus the fictitious downward force of $1/32.2ma$. D'Alembert's principle is stated on page 64.

55. An elevator car has a mass of 1,000 pounds. It gains a velocity upwards of 8 feet per second in 2½ seconds after starting from rest. Calculate (*a*) the tension on the rope while the car is stationary, (*b*) the average tension of the rope while the car is starting upward, and (*c*) the tension of the rope while the car is moving at the full speed of 8 feet per second. Ans. (*a*) 1,000 pounds-weight; (*b*) 1,100 pounds-weight; (*c*) 1,000 pounds-weight.

56a. Find the tension of the rope required in problem 46 to produce an upward acceleration of 2 feet per second per second of the elevator car. Ans. 1596.9 pounds-weight.

Note. The unbalanced upward force necessary to produce the specified acceleration, as calculated by equation (5), is 92.5 pounds-of-force, and the point of application of this upward unbalanced force is the center of mass *C* in Fig. 46*p*. The simplest method of solving this problem is to reduce it to a problem in statics by means of D'Alembert's principle. A downward force of 92.5 pounds may be thought of as acting at the point *C* in addition to the weight of the car, and then the first and second conditions of equilibrium may be applied exactly as in problem 46.

56b. Find the downward acceleration of the car in problem 56*a* when the tension of the rope is 1450 pounds-weight. Ans. 1.032 feet per second per second.

57. A train having a mass of 1,200 tons (2,000 pounds) is to be accelerated at ½ mile per hour per second up a ½ per cent. grade. The train friction is 10 pounds per ton. Find the neces-

sary draw-bar pull of the locomotive. Ans. 79,000 pounds-weight.

Note. A ½ per cent. grade is one that rises ½ foot in 100 feet of horizontal distance.

58. A cord is strung over a pulley. At one end of the cord is a 10 pound body, and at the other end of the cord is a 11 pound body. Neglecting the weight of the cord and the friction and mass of the pulley, find the acceleration of each body and the tension of the cord. Ans. 1.523 feet per second per second; tension of cord 10.476 pounds-weight.

59. A falling ball passes a given point at a velocity of 12 feet per second. How far below the point is the ball after 5 seconds? How far does the ball fall during the fifth second after passing the given point? Air friction neglected. Ans. After 5 seconds the ball is 460 feet below the point, during the fifth second the ball falls 156 feet.

60. A heavy iron ball is tossed at a velocity of 20 feeet per second in a direction 30° above the horizontal. What are its horizontal and vertical distances from the starting point after ¾ second? Air friction neglected. Ans. Horizontal distance 12.99 feet; vertical distance − 1.5 feet.

Note. Find vertical and horizontal components of the initial velocity. The latter component remains unchanged while the vertical motion of the ball is precisely what it would be if it had no horizontal motion.

61. A heavy shot is thrown in a direction 30° above the horizontal, it strikes the ground 50 feet from the thrower, and the shot is 5½ feet above the ground when it leaves the thrower's hand. What is the initial velocity v of the shot? Air friction neglected. Ans. 39.4 feet per second.

Note. The horizontal velocity, $v \cos 30°$, is constant, the time of flight in seconds is $t = 50$ feet $\div (v \cos 30°)$, and the vertical distance fallen, namely 5½ feet, is equal to $v \sin 30 \times t + \frac{1}{2}gt^2$, in which t is the time of flight in seconds. The acceleration of gravity is here to be considered as negative.

62. An 80-ton (2,000 pounds) locomotive goes round a railway curve of which the radius is 600 feet at a velocity of 65 feet per second. With what force in pounds-weight do the flanges

of the wheels of the locomotive push against the outer rail when the outer rail is not elevated? Ans. The flanges of the locomotive exert a horizontal force on the outer rail of 35,210 pounds-weight.

63. Calculate the proper elevation to be given to the outer rail on a railway curve of 600 feet radius for a train speed of 65 feet per second, the width of the track being 4 feet 8½ inches. Ans. 1.015 feet.

64. The tension of a belt is 50 pounds-weight. With what force in pounds-weight does the belt push against each inch of circumference of a pulley 12 inches in diameter when the pulley is stationary? Ans. 8.33 pounds-weight per inch of circumference.

Note. The static relation between tension in a circular hoop and actual outward forces acting on each part of the hoop is the same as the relation between tension and the unbalanced inward forces in the case of a rotating hoop as discussed in Art. 40.

65. The mass of each inch of length of the belt specified in problem 64 is 0.24 pound. With what force in pounds-weight does the belt push inwards against each inch of circumference of the 12-inch pulley when the pulley revolves at a speed of 300 revolutions per minute, the tension of the belt being 50 pounds-weight? Ans. 4.47 pounds-weight per inch of circumference.

Note. The tension of the belt is capable of producing an inward force of 8.33 pounds-weight per inch of circumference of the wheel when the pulley is stationary. When the pulley is rotating the inward force exerted on each inch of circumference of the pulley is reduced by an amount equal to the unbalanced force which must pull inwards on each inch of length of the belt according to equation (10) in Art. 38.

66. The car next to the locomotive in a train is 35 feet long between bumpers and it is pulled at each end with a force of 10,000 pounds (the force at the rear end of the car is of course somewhat less than the force at the front end). The train rounds a circular curve of 1,000 feet radius at a speed of 20 miles per hour. The car with its load weighs 100,000 pounds. Find the horizontal force, at right angles to the track, with which the track acts on the car. Ans. 2,339 pounds-weight.

Note. The portion of a train directly behind the locomotive is under tension like a belt, and the tension helps to constrain the cars to their circular path exactly

as in the case of a belt passing around a pulley. In solving this problem it is sufficiently accurate to use the formula $F = T/r$ in which T is the tension of a belt, r is the radius of the circular arc formed by the belt, and F is the radial force per unit length of belt due to T.

67. A force of 5×10^6 dynes deflects the end of the spring in Fig. 43 through a distance of 1.25 centimeters. What is the value of the constant k in equation (15), and in terms of what unit is this constant expressed? How much force would be required to deflect the end of the spring through a distance of 2 centimeters? Ans. The value of k is 4 million dynes per centimeter; 8 million dynes.

68. A mass of 2 kilograms is attached to the end of the spring specified in problem 67, and the mass is set vibrating. How many complete vibrations will it make per minute? Ans. 427 vibrations per minute.

69. A force of 10 pounds-weight deflects the end of the spring in Fig. 43 through a distance of 0.02 foot. What is the value of the constant k in equation (15) and in terms of what unit is this constant expressed? A mass of 10 pounds is attached to the end of the spring, how many complete vibrations will the 10 pound mass make per minute? Ans. The value of k is 16,000 poundals per foot; 382 vibrations per minute.

70. What is the length l of a simple pendulum which makes one complete vibration per second at a place where the acceleration of gravity is 981 centimeters per second per second? Ans. 24.85 centimeters.

71. A wheel has a mass of 50 pounds, its center of mass is 0.2 inch from the axis of the shaft upon which the wheel rotates, and the speed of the wheel is 600 revolutions per minute. How much force in pounds-weight must act on the shaft to constrain the center of mass to its circular path? What is the direction of the force at each instant? Ans. 102.8 pounds-weight from the center of mass towards the axis of the shaft.

72. A ballistic pendulum AB, Fig. 72p, is suspended by two cords ss, the length of each of which is 400 centimeters, and the body AB has a mass of 10 pounds. A rifle bullet of which the mass

is 0.005 pound, strikes AB at the point indicated by the short arrow, and the velocity imparted to AB carries it through a horizontal distance of 8 inches before it is brought to rest by gravity. Find the velocity of the bullet. The acceleration of gravity is 32 feet per second per second. Ans. 2,264 feet per second.

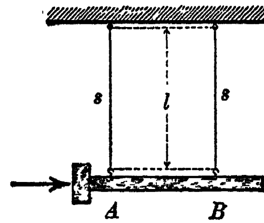


Fig. 72*p*.

Note. The center mass of AB describes the arc of a circle of which the radius is l. Calculate the vertical movement of AB from the known value of l and the specified horizontal movement of AB. Then calculate the velocity of AB which would suffice to lift AB through this vertical distance, and then calculate the velocity of the bullet by using the principle of the conservation of momentum.

73. A ball weighing 550 pounds is shot from a 150,000 pound gun at a velocity of 2,500 feet per second. What is the backward velocity of the gun as the ball leaves the muzzle? Suppose the gun is allowed to move back two feet during the recoil, what is the average value of the force required to bring it to rest? Ans. 9.17 feet per second; 98,540 pounds-weight.

74. An ivory ball of which the mass is 500 grams, and of which the velocity is 100 centimeters per second, collides with a stationary ivory ball of which the mass is 1,000 grams, the line connecting the centers of the balls being parallel to the velocity of the moving ball. Find the common velocity of both balls after their relative motion has been reduced to zero during the first half of the impact, and find the velocity of each ball after impact; specify direction of each velocity. Ans. Common velocity 33.3 feet per second; -33.3 feet per second is the velocity of the small ball after impact, and $+66.7$ feet per second is the velocity of the large ball after impact.

Note. Assume that the ivory balls are perfectly elastic as explained in Art. 47.

CHAPTER V.

FRICTION. WORK AND ENERGY.

51. Friction. A body in motion is always acted upon by dragging forces which oppose its motion and tend to bring it to rest. This action is called *friction.*

Sliding friction. When one body slides on another the motion is opposed by a frictional drag. Thus the cross-head of a steam engine slides back and forth on the guides, a rotating shaft slides in its bearings, and the motion is in each case opposed by a frictional drag.

Fluid friction. The flow of water through a pipe or channel, the motion of a boat, and the motion of a projectile through the air are opposed by friction. This type of friction is called fluid friction and it is discussed in a subsequent chapter.

Rolling friction. The frictional drag upon a wheeled vehicle is due in part to the sliding friction at the journals, in part to the friction of the air, and in part to the continual yielding of the road or track under the wheels. The effect of this yielding is very much as if the vehicle were continually going up a hill, the top of which is never reached. The frictional drag on a wheeled vehicle due to the yielding of the road or track is sometimes called rolling friction.

Frictional drag due to unevenness of a road bed. When a vehicle is drawn very slowly over a rough road, the wheels roll "up hill," as it were, when they strike a small stone and then "down hill" again when they leave the stone, and the *average* value of the pull required to draw the vehicle is not effected by unevenness of road bed; but if the speed of the vehicle is great, the unevenness of the road bed produces a very considerable frictional drag, the effect is as if the wheels were being all the time "rolled up" a succession of small hills not to "roll

down" again, but to come down each time with a bump. This kind of friction shows itself in the vibration and swaying of a vehicle, and it is one of the most prominent causes of frictional drag upon a vehicle which is driven at high speed.

52. Coefficient of sliding friction. The horizontal force H required to cause a body to slide steadily over the smooth horizontal surface of another body is approximately proportional to the vertical force V which pushes the body against the surface. That is

$$H = \mu V \tag{23}$$

in which V is the force with which a body is pushed against any smooth surface, and H is the force, parallel to the surface, which causes the body to slide. The proportionality factor μ is called the *coefficient of friction*; it is nearly independent of the contact area of the sliding substances and it does not vary greatly with the velocity of sliding. Thus the coefficient of friction of wood on a smooth metal surface is about 0.40, the coefficient of friction of smooth brass on smooth steel (not oiled) is about 0.22.

Angle of sliding friction. Consider a block B, Fig. 48*a*, sliding

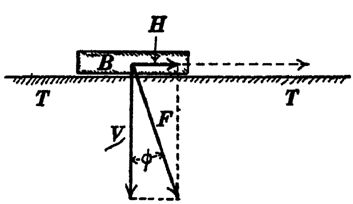


Fig. 48*a*.

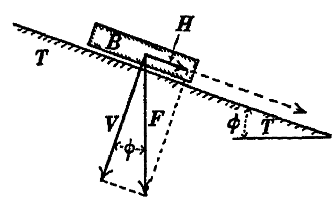


Fig. 48*b*.

on a table TT in the direction of the dotted arrow; let V be the force with which the block is pushed against the table and let H be the force necessary to keep the block in motion. Then since the ratio H/V is constant (that is, if V is large, H is large in proportion), it is evident that the angle φ between V and the resultant force F is constant. This angle is called the *angle of friction* of the given substances B and T, and evidently the tangent of φ is equal to the coefficient of friction μ of the sliding substances.

Figure 48*b* represents the table TT tipped so as to bring the force F into a vertical position. In this case the force F may be thought of as the pull of gravity on the block B, and the component of F parallel to the table (namely H) is barely sufficient to cause the block to slide.

It is important to notice that the force F in Fig. 48*a* is the total force which the sliding block exerts upon the table. Consider blocks AA and BB in contact as shown in Fig. 48*c*. *The block A can exert upon the block B a force in any direction provided the line of action of the force lies inside of a cone of which the half-angle is φ and of which the axis is normal to the surface of contact of AA and BB,* where φ is the angle of friction of the two substances A and B.

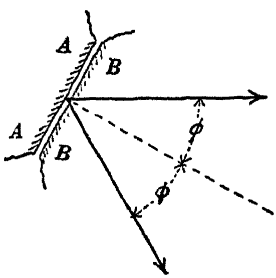


Fig. 48*c*.

The friction between the two sliding surfaces is approximately in accordance with the above statements when the surfaces are smooth and made of unlike materials. Thus wood sliding on metal, polished steel sliding on brass or Babbitt metal, and hard steel sliding on the polished surface of a jewel, all have fairly well defined coefficients of friction. When the sliding surfaces are rough, however, there is no regularity whatever in the friction, and when the substances are similar the friction is sometimes very irregular even though the surfaces are smooth. Thus brass on brass tends to weld and tear in a most remarkable manner and a clean plate of glass cannot be made to slide over another clean plate of glass at all (if the surfaces are very clean) unless there is an air cushion between them.

53. Active forces and inactive forces. Definition of work. Nothing is more completely established by experience than the necessity of employing an active agent such as a horse or a steam engine to drive the machinery of a mill or factory, to draw a car, or to propel a boat; and although the immediate purpose of the

driving force may be described in each case by saying that the driving force overcomes or balances the opposing forces of friction, still the fact remains that the operation of driving a machine or propelling a boat involves a continued effort or cost. Indeed to supply a man with the thing (energy) which will drive his mill or factory, is to supply him with a commodity as real as the wheat he grinds or the iron which he fabricates into articles of commerce. Wheat and iron are sharply defined as commodities in the popular mind on the basis of many generations of commercial activity, because wheat and iron can be stored up and taken from place to place, and because change of ownership is so easily accomplished and so simply accounted for. That which serves to drive a mill or factory, however, cannot be stored up except to a very limited extent, and it is only in recent years that means have been devised for transmitting it from place to place and that an exact system of accounting has been established for governing its exchange. A clear idea of energy does not exist as yet in the popular mind, and the following definitions cannot be expected to convey a full and clear idea at once.

The common feature of every case in which motion is maintained is that *a force is exerted upon a moving body and in the direction in which the body moves.* Such a force is called an *active force,** and to keep up an active force requires continuous effort or cost.

A force which acts on a stationary body, on the other hand, may be kept up indefinitely, without cost or effort; and such a force is called an *inactive force.* Thus a weight resting on a table continues to push downward on the table, a weight suspended by a string continues to pull on the string, the main

*An active force is any mutual force action between two bodies one of which moves with respect to the other. The force with which a boy pulls on a sled is called an active force although there is no relative motion between the boy and the sled. In this case, however, the work is actually done in the boy's legs and the active force in the boy's legs is exerted against the ground which moves backwards with respect to the boy. To push on the front door of a moving car is not to exert an active force.

spring of a watch will continue indefinitely to exert a force upon the wheels of the watch if the watch is stopped.

The idea of an inactive force is applicable also to a force which acts on a moving body but at right angles to the direction in which the body moves. Thus the vertical push of a driver on the seat of a wagon which travels along a level road is an inactive force, the forces with which the spokes of a rotating wheel pull inwards on the rim of the wheel are inactive forces.

An active force is said to do work, and the amount of work done in any given time is equal to the product of the force and the distance that the body has moved in the direction of the force. That is

$$W = Fd \tag{24}$$

in which F is the force acting on a body, and W is the work done by the force during the time that the body moves a distance d in the direction of F. If d is not parallel to F, then $W = Fd \cos \theta$, where θ is the angle between F and d.

Units of work. The unit of work is the work done by unit force while the body, upon which the force acts, moves through unit distance parallel to the force.

The *erg*, which is the *c. g. s.* unit of work, is the work done by a force of one dyne while the body, upon which the force acts, moves through a distance of one centimeter in the direction of the force. The erg is, for most purposes, inconveniently small, and a multiple of this unit, the *joule*, is much used in practice. The *joule** is equal to ten million ergs (10^7 ergs).

The work done by a force of one pound-weight while the body upon which the force acts moves through a distance of one foot in the direction of the force, is called the *foot-pound.*†

*It is frequently convenient to have a name for that unit of force which multiplied by one centimeter gives one joule of work, according to equation (24). This unit of force may be called the joule per centimeter.

†The *kilogram-meter* is the work done by a force of one kilogram-weight while the body upon which the force acts moves through a distance of one meter in the direction of the force: The foot-pound unit of work is used quite generally by American and English engineers, and the kilogram-meter unit of work is used in those countries where the metric system has been adopted.

54. Power. The rate at which an agent does work is called the *power* of that agent. Thus a locomotive exerts a pull of 15,000 pounds-weight on a train and draws the train through a distance of 500 feet in 10 seconds. The work done is 7,500,000 foot-pounds which, divided by the time interval of ten seconds, gives 750,000 foot-pounds per second, as the rate at which the locomotive does work.

Units of power. Power may, of course, be expressed in ergs per second, in joules per second, or in foot-pounds per second. The unit of power, one joule per second is called a *watt.* The *horse-power*, which is extensively used by engineers, is equal to 746 watts or to 550 foot-pounds per second.

Power developed by an active force. Consider a force F acting upon a body which moves in the direction of the force at velocity v. During t seconds the body moves through the distance vt and the amount of work done is $F \times vt$ according to equation (24), and, dividing this amount of work by the time, we have

$$P = Fv \tag{25}$$

in which P is the power developed by an active force F, and v is the velocity with which the body, upon which F acts, moves in the direction of F. If F is expressed in dynes and v in centimeters per second, then P is expressed in ergs per second; if F is expressed in pounds-weight and v in feet per second, then P is expressed in foot-pounds per second.

Example. A horse pulls with a force of 200 pounds weight in drawing a loaded cart at a velocity of 3 feet per second and develops 600 foot-pounds per second of power.

Measurement of power. Nearly all practical measurements relating to work are measurements of power. The power of an agent may be measured as follows:

(*a*) The value of an active force and the velocity of the body upon which the force acts may be measured and the power may then be calculated according to equation (25).

Examples. (1) The draw-bar pull of a passenger locomotive

is measured by means of a heavy spring scale and found to be 6,000 pounds, and the velocity of the locomotive, as determined by the distance traveled in a given time, is found to be 90 feet per second. From these data the net power developed by the locomotive (not counting the power required to propel the locomotive itself) is found to be 540,000 foot-pounds per second, or 991 horse-power.

(2) Let a be the area in square inches of the piston of a steam engine, let p be the average steam pressure in the cylinder in pounds per square inch as measured by a steam-engine indicator, let l be the length of stroke of the piston in feet, and let n be the number of revolutions per second made by the engine. Then the average force pushing on the piston is pa pounds-weight, and the work done during a single stroke is $pa \times l$ foot-pounds, and since the number of single strokes per second is $2n$, the power developed by the steam is $pal \times 2n$, or $2paln$ foot-pounds per second. The power of an engine determined in this way is called its *indicated power*, to distinguish it from the power delivered by the engine to the machinery which it drives. The power delivered by an engine is always less than its indicated power on account of frictional losses in the engine.

(3) An engine to be tested is loaded by applying a brake to its flywheel; the pull on the brake (reduced to the circumference of the fly-wheel) is 200 pounds-weight; the velocity of the circumference of the flywheel, as determined from the measured diameter of the wheel and its observed speed in revolutions per second, is 80 feet per second; and the power developed by the engine is equal to 200 pounds × 80 feet per second, which is equal to 16,000 foot-pounds per second or 29 horse-power. The power of an engine determined in this way is called its *brake power*.

Figure 49 shows the arrangement of a brake for measuring the power of an engine, or of any agent, like an electric motor or water wheel, which delivers power from a pulley. The spring scale S measures the force at the end of the brake arm, and this observed force is multiplied by a/r to find the equivalent force

at the surface of the pulley, where a is the length of the arm as shown in Fig. 49 and r is the radius of the pulley.

(b) Power is frequently measured electrically. Thus the power in watts delivered by a direct-current dynamo is equal to the product of the electromotive force of the dynamo in volts and the current in amperes delivered by the dynamo.

Power-time units of work. Inasmuch as nearly all practical measurements relating to work are measurements of power, it

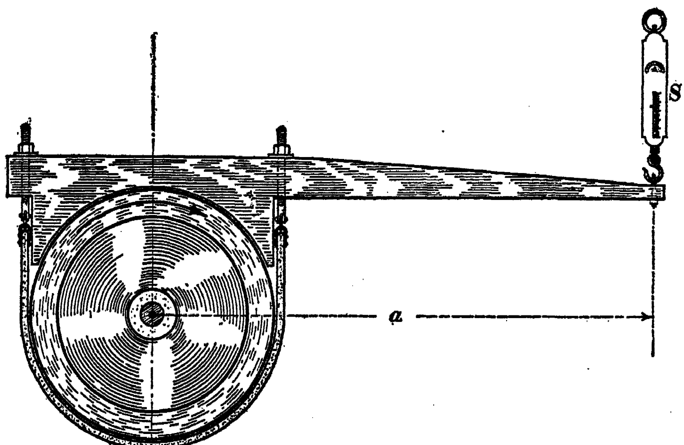


Fig. 49.

has come about that a given amount of work is often expressed as the product of power and time. The *watt-hour* is the amount of work done in one hour by an agent which does work at the rate of one watt, the *kilowatt-hour* is the amount of work done in one hour by an agent which does work at the rate of one kilowatt (one kilowatt is 1,000 watts), and the *horse-power-hour* is the amount of work done in one hour by an agent which does work at the rate of one horse-power.

Efficiency. The efficiency of a machine, like a water wheel, a steam engine, a dynamo, or a motor, which transforms energy, is defined as the ratio of the power developed by the machine to the total power delivered to the machine.

ENERGY.

55. Definition of energy. Limits of the present discussion. Any agent which is able to do work is said to possess *energy*, and the amount of energy an agent possesses is equal to the total work the agent can do. Thus the spring of a clock when it is wound up is in a condition to do a definite amount of work and it is therefore said to possess a definite amount of energy.

In developing the idea of energy it is important to distinguish between an agent which merely transforms energy and an agent which actually has within itself the ability to do a certain amount of work. Thus the steam engine merely transforms the energy of fuel into mechanical work, and a water wheel merely transforms the energy of an elevated store of water into mechanical work, whereas a clock spring, when wound up, has a store of energy within itself.

Whenever a substance or a system of substances gives up energy which it has in store, the substance or system of substances always undergoes change. Thus the fuel which supplies the energy to a steam engine and the food which supplies the energy to a horse, undergo *chemical change;* the steam which carries the energy of the fuel from the boiler to the engine *cools off* or undergoes a *thermal change* when it gives up its energy to the engine; a clock spring *changes its shape* as it gives off energy; an elevated store of water *changes its position* as it gives off energy; the heavy fly wheel of a steam engine does the work of the engine for a few moments after the steam is shut off and the fly wheel *changes its velocity* as it gives off its energy.

Not only does a substance undergo a change when it *gives up energy* by *doing work*, but a substance which *receives energy* or has *work done upon it* undergoes a change. Thus when air is compressed by a bicycle pump, work is done on the air and it becomes warm; the work done in keeping up the motion of any machine or device produces heat at the places where friction occurs; when a clock spring is wound up it stores energy by its change of shape; when water is pumped into an elevated tank

it stores energy by its change of position; a large part of the work which is expended on a heavy railway train at starting is stored in the train by its change of velocity.

We are now facing a very important question; shall we attempt a complete discussion of the whole theory of energy at once by examining into all kinds of changes which take place when a substance does work or has work done upon it; or shall we base our discussion on one thing at a time? Most assuredly the latter. Therefore let us proceed to discuss the energy relations involved in purely mechanical changes, namely, changes of position, changes of velocity, and changes of shape,* and let us exclude everything else from our present discussion such as chemical changes and thermal changes.

In attempting to exclude thermal changes from our present discussion, however, we are confronted by the fact that friction (with its accompanying thermal changes) is always in evidence everywhere; and it requires a very high degree of analytical power to think only of purely mechanical changes in the face of such a fact. This necessary feat of mental effort is greatly facilitated by the use of the idea of a *frictionless system;* and this term will be used whenever it is desired to direct the reader's attention exclusively to the energy relations that are involved in purely mechanical changes.

Before proceeding to a minute examination into the mechanical theory of energy, it is desirable to establish the ideas of *kinetic energy* and *potential energy* on the basis of general experience. Suppose that a post, standing beside a railway track, is to be pulled out of the ground; can a car-load of stone be made to do the work? Certainly it can. All that is necessary is to have the car moving past the post and to throw over the post a loop of cable which is attached to the moving car. A moving car is able to do work; and when it does work its velocity is reduced, and its store of energy decreased. The energy which a body

*Changes of shape are discussed in Chapter VII.

stores by virtue of its velocity is called the *kinetic energy* of the body.

It is also a familiar fact that a weight can drive a clock, but in doing so the position of the weight changes and its store of energy is reduced. The energy which a body stores by virtue of its position is called the *potential energy* of the body.

The physical reality which lies behind the terms kinetic energy and potential energy can perhaps be shown most clearly by considering a bicycle rider. Suppose that the rider faces a steep hill or a sandy stretch of road where he is called upon to do an unusual amount of work. Every bicycle rider realizes the advantage of having a large velocity in such an emergency. This *advantage of velocity* is called kinetic energy.* Or suppose that a bicycle rider wishes to use his whole strength, or more if he had it, in covering a certain distance; every bicycle rider realizes the advantage of being on top of a hill in such an emergency; this *advantage of position* is called potential energy.

56. Kinetic energy of a particle. The kinetic energy of a particle is given by the equation

$$W = \tfrac{1}{2}mv^2 \tag{26}$$

in which W is the kinetic energy in ergs (or foot-poundals), m is the mass of the particle in grams (or pounds), and v is its velocity in centimeters per second (or in feet per second).

Proof of equation (26). The kinetic energy of a particle may not only be defined as the work it can do when stopped, but also as the work required to establish its motion. Let a constant unbalanced force F act upon a particle of mass m, then

$$F = ma. \tag{i}$$

After t seconds the velocity gained is

$$v = at \tag{ii}$$

*Of course a body can have velocity only in relation to another body and the idea of kinetic energy is an idea which applies strictly to a system of particles but not to an individual particle. The velocity in equation (26) is velocity referred to the earth.

and the distance traveled is

$$d = \tfrac{1}{2}at^2. \qquad \text{(iii)}$$

as explained in Art. 35. Therefore, multiplying equations (i) and (iii), member by member, we have

$$Fd = \tfrac{1}{2}ma^2t^2, \qquad \text{(iv)}$$

but Fd is equal to the work done on the particle and a^2t^2 is equal to v^2, according to equation (ii), so that equation (iv) reduces to $W = \tfrac{1}{2}mv^2$.

The kinetic energy of a system of particles is, of course, equal to the sum of the kinetic energies of the individual particles of the system.

When mass is expressed in pounds and velocity in feet per second, then the kinetic energy of a particle in foot-pounds is given, approximately, by the equation

$$W = \frac{1}{64.4}mv^2. \qquad (27)$$

57. Potential energy. The energy stored in a system by virtue of the configuration of the system, that is, by virtue of the relative positions of the parts of the system, is called the *potential energy* of the system. For example a weight stores energy by virtue of its position relative to the earth; a bent spring stores energy by virtue of its elastic distortion.

It is impossible to assign a definite amount of potential energy to a system which has a given configuration, for it is impracticable to assign a definite limiting configuration beyond which the system cannot go. Thus the weight of a clock might have its *available* store of potential energy increased by boring a hole in the clock case so that the weight would move down to the floor, then a hole could be bored in the floor and eventually a deep well could be dug in the ground. In order to be able to speak definitely of the potential energy of a weight it is necessary, therefore, to assign an arbitrary *zero position* and to reckon the poten-

tial energy of any other given position as the work the weight can do in changing from the given position to the chosen zero position.

In general the potential energy of any system in a given configuration may be defined as the amount of work the system can do in changing from the given configuration to an arbitrarily chosen *zero configuration*.

Conservative systems. A system (frictionless) which does the same amount of work in passing from one configuration to another, whatever the intermediate stages may be through which the system passes, is called a *conservative system*, **and the idea of potential energy applies only to such systems.** Suppose, for example, that a weight would do more work in moving from a given position to its chosen zero position over one path A than over another path B, see Fig. 50*a*; then the potential energy of the weight in the given position would be indefinite; and if the weight were carried around the closed path AB in the direction of the arrows, then a large amount of work would be done in passing down path A and only a *portion* of this work would be required to carry the weight back to the given position over path B. That is, work would be created every time the weight completed the cycle of motion around AB, and we would have "perpetual motion," that is a machine which would do work without suffering any permanent change of any kind.* *All physical systems are conservative in so far as purely mechanical changes are concerned;* and experience shows that all physical systems are conservative when changes of all kinds, mechanical, chemical, thermal and electrical, are taken into account; that is, the

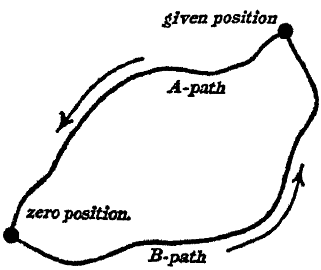


Fig. 50*a*.

*The idea involved in this discussion of Fig. 50*a* may be strengthened by introducing the idea of cheating to which it stands in clear apposition. Suppose one were to hold a weight in his hand and allow it to move downwards in full view of a class, and then bring it again to its former position by passing it behind his back where it is out of sight with the idea of avoiding the doing of work!

energy that a system gives off when it undergoes any change whatever, depends only upon the initial and final states of the system, and is independent of the intermediate stages through which the system may be made to pass.

Perpetual motion impossible. A perpetual motion machine would be a device which would furnish a continuous supply of energy for driving machinery. Most of the attempts to produce perpetual motion have been quite ridiculous, but on the other

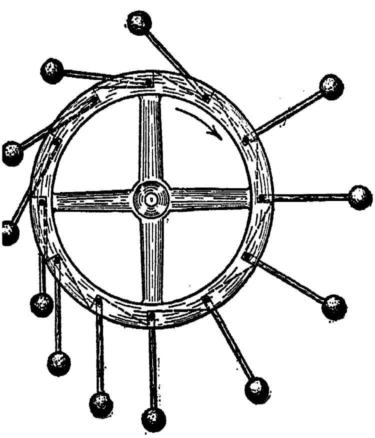


Fig. 50*b*.

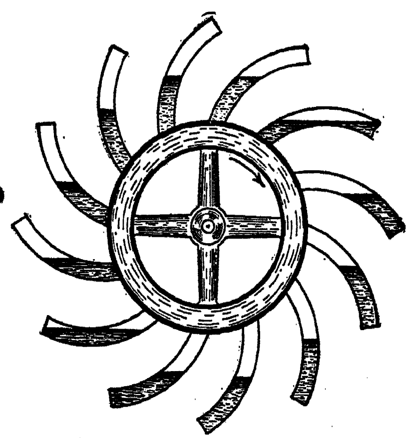


Fig. 50*c*.

hand many attempts have been quite reasonable. The reasonable attempts have nearly all been attempts to get more work out of a weight while it falls along one path than is required to carry the weight back to its starting point along another path. Figures 50*b* and 50*c* show two perpetual motion devices which were proposed and tried about 1750. Figure 50*b* is a rachet wheel to which a number of hinged arms are attached, each arm carrying a heavy weight. Fig. 50*c* is a wheel to the rim of which a number of bent tubes are attached, each tube containing mercury. The arrows show the directions in which the wheels were expected to be driven by the increased leverage of the falling weights or of the falling mercury.

58. Mutual relation between the kinetic energy and the potential energy of a closed system. The most clearly intelligible statement of the idea as to the impossiblity of perpetual motion may be made by considering an ideal system upon which no forces act from the outside. During the motion of such a system no work would be done upon the system by outside agents and no work would be done by the system upon any outside body. Such an ideal system is called a *closed system*. The sun and planets constitute sensibly a closed system in so far as their mechanical actions and reactions are concerned, that is to say, the motion of the sun and planets is modified by the forces which they exert on each other but not perceptibly modified by the forces which are exerted upon them by the distant stars.

Consider a closed system (the solar system for example) the particles* of which are in motion, and let us consider what takes place in any short interval of time. In the first place, each particle moves through a certain small distance, the configuration of the system is changed accordingly, and *the potential energy of the system decreases* (or increases) by an amount which is equal to the work done on all the particles by their mutual force actions. In the second place, the velocity of each particle is changed by the forces acting upon it, the kinetic energy of each particle increases (or decreases) by an amount equal to the work done upon it, and *the total kinetic energy of the system increases* (or decreases) by an amount which is equal to the work done on all the particles by their mutual force actions. Therefore the decrease (or increase) of potential energy of a closed system is always equal to the accompanying increase (or decrease) of the kinetic energy of the system, or in other words, the sum of the potential and kinetic energies of a closed system is constant.

59. The principle of the conservation of energy. It may appear from the argument of the preceding article that the total energy of a closed system must always be constant. *This indeed*

*The individual planets and even the sun may be considered as particles in so far as their action on each other is concerned because they are so far apart.

is true if the idea of potential energy is a legitimate idea, that is to say, if the work done by the mutual force actions between the particles of the system when the system is changed from one configuration to another configuration is independent of the intermediate stages through which the system is made to pass, or in other words, if the system is what is called a conservative system. This constancy of the total energy of a closed system of the kind specified (a conservative system, indeed all systems are conservative so far as known) is called *the principle of the conservation of energy* and reduced to its simplest terms it is that **the work done by a system depends only upon the initial and final states of the system and it is hopeless to seek a roundabout method for bringing the system back to its initial state by a smaller expenditure of work.**

The usual statement of the principle of the conservation of energy is that *energy can neither be created nor destroyed;* but this statement is so completely abstracted from actual physical considerations that it is almost meaningless.

60. Application of the principle of work to statics. The principle of virtual work. Consider a body which is acted upon by a number of forces. If the body were to be given any slight dis-

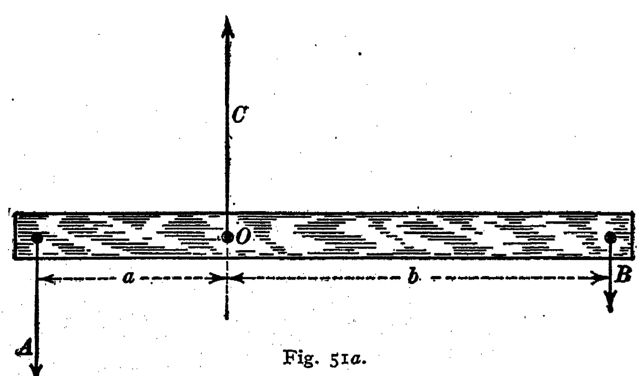


Fig. 51*a*.

placement whatever a certain total amount of work (called *virtual work*) would be done by the forces. *This virtual work is equal to zero when the forces are in equilibrium.* Thus, Fig. 51*a* represents a body in equilibrium under the action of the three forces A, B

and C. If the body were moved a small distance upwards, a small amount of work would be done *on the body* by the force C, and an equal amount of work would be done *by the body* on the agents which exert the forces A and B, that is to say, the total work which would be done *on the body* by the forces A, B and C during the displacement would be equal to zero.

Proof of the parallelogram of forces. Consider two forces A and B which act upon a particle O as shown in Fig. 51*b*. The resultant R of these forces may be

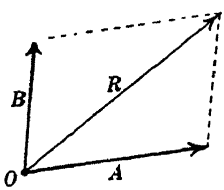


Fig. 51*b*.

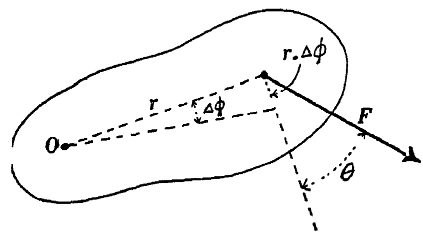


Fig. 51*c*.

defined as that force which would do the same amount of work as the forces A and B together during any small displacement whatever of the point O. Imagine the point O to be displaced a distance Δx along the x-axis of reference and let X_a and X_b be the x-components of the given forces A and B, respectively. The work done by the two forces A and B during the displacement Δx would be $(X_a + X_b) \cdot \Delta x$. Let X_r be the x-component of the force R. The work done by R during the displacement Δx would be $X_r \cdot \Delta x$. Therefore placing these two expressions of virtual work equal to each other, we have

$$X_r \cdot \Delta x = (X_a + X_b) \cdot \Delta x$$

or

$$X_r = X_a + X_b \qquad \text{(i)}$$

That is to say, the x-component of the resultant of two forces is the sum of the x-components of the two forces and a similar statement applies to y-components. Now the projection of the diagonal of any parallelogram on an axis of reference is equal to the sum of the projections of the two adjacent sides of the parallelogram on that axis. Therefore equation (i) means that the resultant of two forces is represented by the diagonal of a parallelogram of which the sides represent the given forces. This argument proves that the resultant of A and B is equal and parallel to R. To show that it is colinear with R, consider that the two given forces A and B have zero torque action about the point O so that the single force which is equivalent to A and B combined must have zero torque action about the point O and it must therefore pass through the point O.

Definition of torque. The torque action of a force about a given axis may be *defined* as that factor by which a slight angular displacement about the axis must be multiplied to give the work done by the force during the displacement. Thus, Fig. 51*c* represents a body pivoted on an axis O and displaced through a small angle $\Delta\phi$ about O. The point of application of the force F moves through a dis-

tance $r \cdot \Delta\phi$, and the work done by F is $Fr \cdot \Delta\phi \times \cos\theta$. That is, the factor by which the angular displacement $\Delta\phi$ is multiplied to give the work done is $Fr \cdot \cos\theta$, and this is the measure of the torque action of the force F about the axis O.

Example 1. *Application of principle of virtual work to the lever.* Let the body shown in Fig. 51*a* be turned about the point O (the fulcrum) through the angle $\Delta\varphi$ thus causing the point of application of force A to move downwards through the distance $a \cdot \Delta\varphi$, and the point of application of force B to move upwards through the distance $b \cdot \Delta\varphi$; then $Aa \cdot \Delta\varphi$ is the work done on the body by force A, and $Bb \cdot \Delta\varphi$ is the work done by the body on the agent which exerts force B. Therefore $Bb \cdot \Delta\varphi$ is to be considered as negative work done on the given body, so that the total work done on the body by the forces A and B during the given displacement is $Aa \cdot \Delta\varphi + Bb \cdot \Delta\varphi$ and this is equal to zero if the forces are in equilibrium. Therefore we have

$$Aa \cdot \Delta\varphi + Bb \cdot \Delta\varphi = 0$$

or

$$Aa + Bb = 0$$

which is the well-known equation of the lever.

Example 2. *Application of the principle of virtual work to a barrel hoop.* Figure 52 represents a hoop which can be tightened by means of the bolt b. Imagine the bolt b to be shortened by a certain very small amount l. The work done in thus shortening the bolt would be Tl, where T is the force exerted by the bolt on either of the flanges, that is to say, T is the tension of the hoop; but to shorten the bolt by the amount l would shorten the radius of the hoop by the amount $l/2\pi$, so that each unit of circumference of the hoop would move inwards through a distance

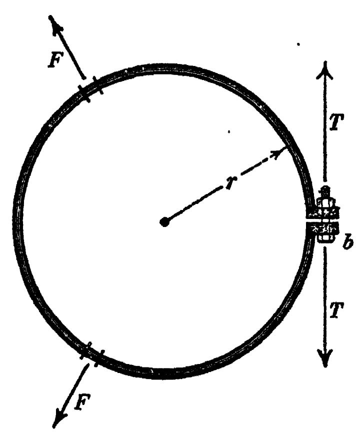


Fig. 52.

$l/2\pi$. Let F be the outward force exerted by the barrel on each unit of length of the hoop (or the inward force exerted on the barrel by each unit of length of hoop), then $F \times l/2\pi$ would be the work done by each unit length of hoop during the shortening of the bolt, and the total work done by the entire hoop would be $F \times l/2\pi \times 2\pi r$, where r is the radius of the hoop, Placing this work done by the hoop equal to the work done on the bolt we have

$$Tl = F \times \frac{l}{2\pi} \times 2\pi r$$

or

$$T = Fr \quad .$$

in which T is the tension of a hoop, r is the radius of the hoop, and F is the force with which each unit of circumference of the hoop pushes inwards on the barrel. In this discussion the stiffness of the hoop is ignored.

PROBLEMS.

75. A 165-pound man climbs a height of 40 feet in 11 seconds. How much work is done, and at what rate? Express the work in foot-pounds, and in joules; and express the power in foot-pounds per second, in horse-power, and in watts. Ans. 6,600 foot-pounds or 8,940 joules of work done at the rate of 600 foot-pounds per second, or 814 watts, or 1.09 horse-power.

76. A horse pulls upon a plow with a force of 100 pounds weight and travels 3 miles per hour. What power is developed? Express the result in foot-pounds per second, in watts, and in horse-power. Ans. 440 foot-pounds per second; 597 watts; 0.8 horse-power.

77. A belt traveling at a velocity of 70 feet per second transmits 360 horse-power. What is the difference in the tension of the belt between the tight and loose sides in pounds weight? Ans. 2,829 pounds-weight.

Note. When a belt drives a pulley the tension of the belt is greater on one side than on the other. Let F be the tension of the belt on the tight side and F' the tension of the belt on the loose side, and let v be the velocity of the belt. The

rate at which work is done on the driven pulley by the tight side of the belt is Fv and the rate at which work is delivered back to the driving pulley by the loose side of the belt is $F'v$. Therefore the net rate at which work is delivered to the driven pulley is $(F-F')v$.

78. A stream furnishing 500 cubic feet of water per second falls a distance of 15 feet. What power can be developed from this stream by a water wheel of which the efficiency is 60% ? Ans. 511 horse-power.

79. The engines of a steamship develop 20,000 horse-power, of which 30 per cent. is represented in the forward thrust of the screw in propelling the ship at a speed of 17 miles per hour. What is the forward thrust of the screw in pounds-weight? Ans. 132,350 pounds-weight.

Note. The useful part of the power developed by the engines of a steamship is represented by the forward thrust of the propellor shaft against the framework of the ship, and the useful power is equal to the product of this force times the velocity of the ship.

80. An electric motor has an efficiency of 80 per cent. and electrical energy costs 5 cents per kilowatt-hour. How much does the output of the motor cost per horse-power hour? Ans. 4.66 cents.

81. A 1,000 horse-power boiler and engine plant costs about $70,000 complete, including land, building, boilers, engines and auxiliary apparatus such as pumps and feed water heaters. The cost of operating this plant continuously, night and day, is as follows:

Interest on investment	5 per cent. per annum.
Depreciation	10 " " " "
Maintenance and repairs	4 " " " "
Taxes and insurance	2 " " " "

Labor $30 per day, 365 days in year.
Coal $2.00 per ton.

The average demand for power is 50 per cent. of the rated power output of the plant, that is 500 horse-power, and the consumption of coal is $2\frac{1}{2}$ pounds per horse-power-hour. Find the cost of a horse-power-hour delivered by the engine. Ans. 0.83 cent.

82. The above engine will drive a 700 kilowatt dynamo, that is a dynamo capable of delivering 700 kilowatts, The cost of dynamo, station wiring and switch-board apparatus is $20,000. The average output of the dynamo is 350 kilowatts (corresponding to 500 horse-power output of engine). Calculate the cost of electrical energy per kilowatt-hour at the station, allowing 21 per cent. for interest, depreciation, etc., on the electrical machinery and allowing $5 per day additional for labor. Ans. 1.38 cents.

83. A steam engine indicator shows an average steam pressure of 55 pounds per square inch (reckoned above atmospheric pressure) during each stroke of a steam engine, and the engine exhausts into a condenser where the pressure is 13 pounds per square inch below atmospheric pressure. The diameter of the piston is 16 inches, the diameter of the piston rod is 3 inches, the length of stroke is 24 inches, and the engine makes 75 revolutions per minute. Find the indicated horse-power of the engine. Ans. 119.9 horse-power.

Note. In solving this problem consider that the piston rod projects out of both ends of the cylinder so that the steam acts upon the piston surface which is outside of the piston rod on both strokes.

84. A brake test of a steam engine gave the following data: speed of engine 200 revolutions per minute, length of brake arm (*a*, Fig. 49) 7½ feet, observed force at end of brake arm and at right angles to arm 240 pounds-weight. Find the brake horse-power of the engine. Ans. 68.6 horse-power.

Note. Reduce the force at the end of the brake arm to the equivalent force at the circumference of the pulley, and consider that the rim of the pulley is being moved in opposition to this force at a known velocity, so that the work done in driving the pulley is equal to the product of the force times the velocity of the rim of the pulley.

85. A fan blower is mounted on a cradle which swings on knife edges in the line of the axis of the fan. When the fan is driven, the cradle tends to tip to one side and this tendency is balanced by a weight sliding on a horizontal lever arm, as shown in Fig. 85*p*. The belt is thrown off the fan, the sliding weight moved to give a balance and the "zero position" of the weight is observed.

The fan is then driven at a speed of 1,800 revolutions per minute and the weight (10 pounds) has to be moved $6\frac{3}{4}$ inches from its zero position to balance the driving torque exerted by the belt on

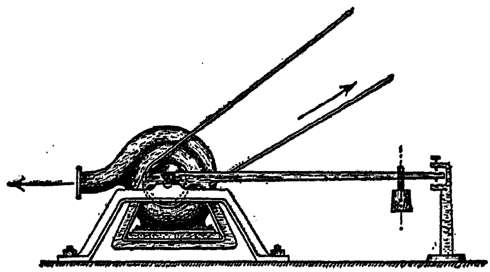


Fig. 85*p*.

the fan. Find the power expended in driving the fan and express it in horse-power and in watts. Ans. 1.93 horse-power; 1,440 watts.

Note. Let r be the radius of the pulley. Then the velocity of the belt is known in terms of r. Let F be the tension of the belt on the tight side and F' the tension of the belt on the loose side. Then $(F-F')r$ is the total torque action exerted on the fan by the belt, and this torque is balanced by moving the 10-pound weight through a distance of $6\frac{3}{4}$ inches. Therefore from the equation

$$(F - F')r = \frac{6.75}{12} \times 10 \text{ pound-feet}$$

the value of $F-F'$ can be determined in terms of r.

When the velocity of the rim of the pulley (the velocity of the belt) is multiplied by $F-F'$ the unknown radius r cancels out, and the power delivered to the fan is completely determined.

86. Four idle pulleys A, B, C and D, Fig. 86*p*, are mounted in a frame which is free to rotate about the point O which is the point of intersection of the left hand stretches of belt. A weight W slides along a lever arm which is fixed to the rocking frame so that the tilting action of the right hand stretches of belt may be balanced and measured. When no power is transmitted by the belt, the tension of the belt is the same everywhere and, under these conditions, the weight W is adjusted to its zero position to give a balance. When power is transmitted to a given machine the belt tensions F' and F'' differ and the weight W is moved farther

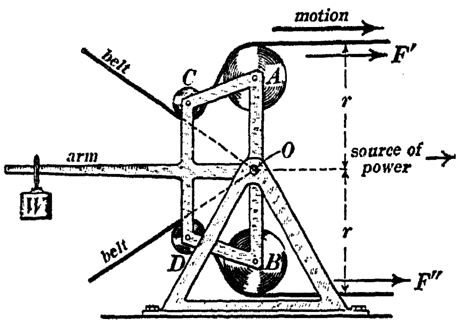


Fig. 86*p*.

away from O to give a balance. This movement of the weight W from its zero position is 16 inches, the mass of W is 55 pounds, the distance r is 24 inches, and the speed of the belt is 80 feet per second. Find the power transmitted by the belt. Ans. 5.33 horse-power.

Note. To find the value of $F'-F''$ consider the balance of torque actions about the point O.

87. A shaft transmits 100 horse-power and runs at a speed of 250 revolutions per minute. Calculate the torque exerted on the shaft. Express the result in pound-feet, in pound-inches, and in dyne-centimeters. Ans. 2,100 pound-feet; 25,200 pound-inches 284×10^8 dyne-centimeters.

Note. The simplest argument of this problem is to imagine the shaft to be cut, and then coupled by a device shown in Fig. 87*p*. The required torque is equal

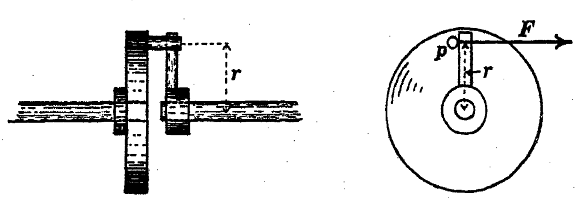


Fig. 87*p*.

to the product of the force F times the arm r, and the power which is transmitted is equal to the product of the force F times the velocity of the pin p

88. A steamship has a gross mass of 25,000 tons. What is the kinetic energy of the ship at a speed of 18 miles per hour?

Express the result in foot-pounds and in horse-power-hours. Ans. 544.5×10^6 foot-pounds; 275 horse-power-hours.

89. A bicycle rider has a 50-foot hill to climb. What velocity must he have at starting to relieve him from the doing of one third of the work required? Ans. 32.7 feet per second.

90. The rim of the fly-wheel of a metal punch is 5 feet in diameter and its mass is 560 pounds. At what initial speed must the fly-wheel run in order that the punch may exert a force of 72,000 pounds through a distance of one inch and reduce the speed of the fly-wheel only 30 per cent? Ans. 2.33 revolutions per second.

91. A counterpoise of ¾ pound balances a weight of 100 pounds wherever the weight may be placed on the platform of a balance scale. In what way and to what extent does the platform move when the counterpoise moves ½ inch downwards? Ans. Every part of the platform moves 0.00375 inch upwards.

Note. This problem may be solved in the simplest possible manner by using the principle of virtual work.

92. A screw-jack is turned by a lever of which the radial length is 18 inches, and the pitch of the screw is ⅜ inch. What is the lifting force produced by a pull of 100 pounds on the end of the lever, neglecting friction? Ans. 30,160 pounds-weight.

Note. Consider the distance travelled by the end of the lever and the travel of the screw in one complete turn, and apply the principle of virtual work.

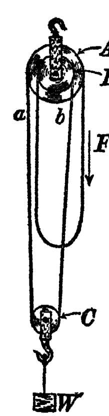


Fig. 93*p*.

93. The differential pulley consists of a large pulley *A* and a smaller pulley *B* made in one piece, and a third pulley *C*, all threaded with an endless chain as shown in Fig. 93*p*. The pulleys *A* and *B* are sprocket wheels with notches which engage the links of the chain so that the chain cannot slip on *A* and *B*. One turn of *A* and *B* takes in at *a* a length of chain which is equal to the circumference of the larger pulley *A* and pays out at *b* a length of chain which is equal to the circumference of the smaller pulley *B*.

The circumference of A contains 12 notches, the circumference of B contains 11 notches and the length of each link of the chain is 1 ½ inches. What is the lifting force produced by a pull of 150 pounds at F, neglecting friction? Ans. 3,600 pounds-weight.

Note. When the point of application of the force F in Fig. 93*p* moves downward through a distance x the weight W is lifted through a distance y. The values of x and y may be easily found from the data of the problem.

94. A wooden tank 10 feet in diameter is held together by hoops of steel. The tension in one of the hoops is 1,000 pounds of force. Find the force exerted against the tank by each foot-length of hoop. Ans. 200 pounds of force.

CHAPTER VI.

ROTATORY MOTION.

61. Rotation about a fixed axis. Definitions. The simplest case of rotatory motion is that which is exemplified by the rotation of a wheel about a fixed axis. We shall first consider this simple case in detail and then proceed to the more complicated rotation about a moving axis. In order to rivet the attention to rotatory motion to the exclusion of movements of distortion, the idea of a *rigid body* will be used throughout the chapter, a rigid body being an ideal body which cannot change its shape or size.

Angular velocity. Let φ be the angle turned by a rotating body during t seconds; the quotient φ/t is called the *average angular velocity* of the body during the t seconds. If the time interval is very short, the quotient $\Delta\varphi/\Delta t$ is the actual angular velocity of the body at the given instant, $\Delta\varphi$ being the angle turned by the rotating body during the short time interval Δt. When the angle φ is expressed in radians and time t in seconds, then the quotient φ/t is in *radians per second.* Angular velocity is expressed in radians per second throughout this chapter. In practice, angular velocity is generally expressed in revolutions per second. There are 2π radians in one revolution, and therefore one revolution per second is equal to 2π radians per second, or, in general

$$\omega = 2\pi n \tag{28}$$

in which ω is the angular velocity of a body in radians per second and n is the angular velocity in revolutions per second.

Angular velocity is frequently called *spin-velocity* or, simply, *spin.*

Angular acceleration. In many machines a part may rotate at a variable angular velocity. This is most strikingly illustrated

by the motion of the balance wheel of a watch. The rate of change of the angular velocity of a body is called its *angular acceleration*. Thus an engine is started, and after six seconds the fly wheel has an angular velocity of 4 revolutions per second (= 25.13 radians per second), so that the average angular acceleration of the wheel during the six seconds is 4.1888 radians per second per second. Of course, the fly wheel may have gained most of its angular velocity during a portion of the six seconds, so that 4.1888 radians per second per second is merely its average angular acceleration. The angular acceleration of a rotating body at a given instant is equal to the quotient $\Delta\omega/\Delta t$ where $\Delta\omega$ is the angular velocity gained during the short interval of time Δt.

Angular acceleration is frequently called *spin-acceleration.*

62. Unbalanced torque and angular acceleration. Definition of moment of inertia. When a wheel is set in rotation, an unbalanced torque must act upon the wheel. This is exemplified in the operation of spinning a top. When a rotating wheel is left to itself it loses its angular velocity and comes to rest on account of the friction of the wheel against the air and on account of the

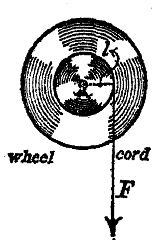


Fig. 53*a*.

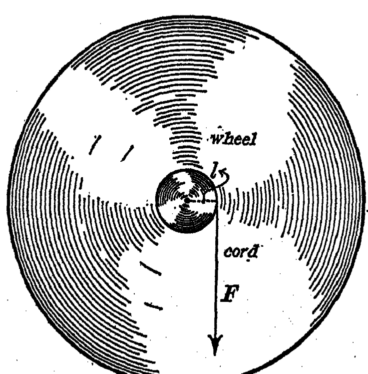


Fig. 53*b*.

friction of the shaft in its bearings. To maintain a steady motion of rotation of a wheel, a driving torque must act upon the wheel sufficient to balance the opposing torque due to friction.

The effect of an unbalanced torque is to change the angular velocity of a wheel, or, in other words, or produce angular acceleration, positive or negative as the case may be. The angular acceleration of a given wheel is proportional to the unbalanced torque which acts upon the wheel, and a given unbalanced torque produces a small angular acceleration of a large heavy wheel, or a large angular acceleration of a small light wheel. Thus, if a cord be wrapped around the shaft upon which a wheel is mounted, a pull on the cord produces torque equal to Fl, see Figs. 53*a* and 53*b*; and this torque imparts angular velocity to the small light wheel, Fig. 53*a*, at a rapid rate, whereas it imparts angular velocity to the large heavy wheel, Fig. 53*b*, at a much slower rate.

When a wheel is rotating every particle of the wheel moves at a definite linear velocity, and when the angular velocity of the wheel increases it is evident that the linear velocity of every particle of the wheel must increase; that is to say, *angular acceleration* of a wheel involves *linear acceleration* of every particle in the wheel, and it is possible to show the exact relation between angular acceleration and the unbalanced torque which produces it, by considering the linear acceleration of each particle in a wheel. The following discussion of this matter is the foundation of the dynamics of rotatory motion and it leads to a definition of what is called the *moment of inertia* of a wheel.

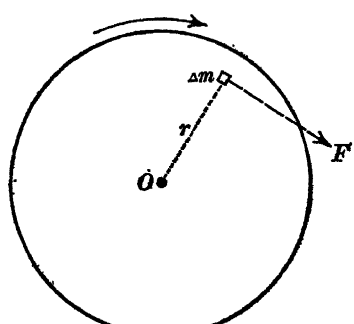


Fig. 54.

Figure 54 represents a wheel rotating n revolutions per second, or $2\pi n$ radians per second, about the axis O. The particle Δm describes a circular path of which the circumference is $2\pi r$, the particle traces this circumference n times per second, and therefore the linear velocity v of the particle is $2\pi rn$ centimeters per second, r being expressed in

centimeters; but $2\pi n$ is equal to the angular velocity ω of the wheel in radians per second, and, therefore

$$v = r\omega \tag{29}$$

If the angular velocity of the wheel is changing, the linear velocity v of the particle m must change r times as fast as ω, inasmuch as v is always r times as large as ω. Therefore, representing the angular acceleration of the wheel by α (rate of change of ω) and representing the linear acceleration of the particle by a* (rate of change of v) we have

$$a = r\alpha \tag{30}$$

In order to produce the acceleration a of the particle, an unbalanced force F, see Fig. 54, must act on the particle in the direction of a, this force, expressed in dynamic units, must be equal to $\Delta m \cdot a$ according to equation (1) Art. 33, and the torque action of this force is equal to $Fr(= \Delta m \cdot a \times r)$; but a is equal to $r\alpha$ according to equation (30), so that $Fr = \Delta m \cdot r^2\alpha$, or representing Fr by ΔT, we have

$$\Delta T = \alpha r^2 \cdot \Delta m$$

in which ΔT is that part of the unbalanced torque T acting on the wheel which causes the linear acceleration of the given particle Δm. Consider in this way all of the particles of the wheel and we have

$$\Delta T = \alpha r^2 \cdot \Delta m$$

$$\Delta T_1 = \alpha r_1^2 \cdot \Delta m_1$$

$$\Delta T_2 = \alpha r_2^2 \cdot \Delta m_2$$

$$\Delta T_3 = \alpha r_3^2 \cdot \Delta m_3$$

etc., etc., whence, by adding, we have

*We are not concerned here with the *radial acceleration* of the particle Δm, since the radial acceleration is produced by unbalanced *radial forces* which have no torque action about O. Radial accelerations of the particles of a wheel have nothing to do with the angular acceleration of the wheel.

$$T = \alpha(r^2 \cdot \Delta m + r_1^{\ 2} \cdot \Delta m_1 + r_2^{\ 2} \Delta m_2 + \cdots)$$

or

$$T = \alpha \Sigma r^2 \cdot \Delta m$$

or writing

$$K = \Sigma r^2 \cdot \Delta m \tag{31}$$

we have

$$T = K\alpha \tag{32}$$

The quantity K, which is obtained by multiplying the mass of each particle of the wheel by the square of its distance from the axis and adding all of these products together, is called the *moment of inertia* of the wheel, and equation (32) shows that the unbalanced torque acting on a wheel is equal to the product of the moment of inertia of the wheel and the angular acceleration of the wheel.

Units involved in equations (31) and (32). If c.g.s. units are used throughout, then moment of inertia is expressed in grams $\times$ centimeters squared (gr. cm.^{2}), torque is expressed in dynes $\times$ centimeters and, of course, angular acceleration is expressed in radians per second per second. Equations (31) and (32) hold good, however, when moment of inertia is expressed in pounds $\times$ feet squared (lb. ft.^{2}), torque in poundals $\times$ feet and angular acceleration in radians per second per second. If torque is expressed in pounds-weight $\times$ feet, moment of inertia in pound-feet-squared and angular acceleration in radians per second per second, then equation (32) becomes

$$T = \frac{1}{32.2} K\alpha \tag{33}$$

approximately.

Example of the calculation of moment of inertia. The moment of inertia of a homogeneous solid of regular form can be calculated by the methods of calculus. Consider, for example, a long slim rod of length L and mass M rotating about its middle point O as shown in Fig. 55. The mass of the short portion dr is $(M/L) \times dr$ and its distance from O is r. Therefore, writing $(M/L) \times dr$ for Δm in equation (31) we have

$$K = \frac{M}{L} \Sigma r^2 dr$$

but the sum (integral) $\Sigma r^2 dr$ between the limits $r = +L/2$ and $r = -L/2$ is equal

to $\frac{1}{12}L^3$, so that $K=\frac{1}{12}ML^2$. The moments of inertia given in the following table were calculated in this way.

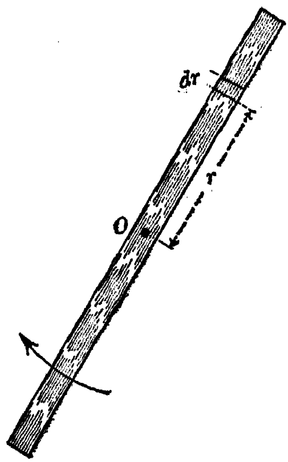


Fig. 55.

Radius of gyration. The radius of gyration of a rotating body is the distance ρ from the axis of rotation at which the entire mass M of the body might be concentrated without altering the moment of inertia of the body. If the entire mass M were concentrated at distance ρ from the axis, the moment of inertia would be equal to $M\rho^2$, according to equation (31). That is

$$K = M\rho^2 \tag{34}$$

or

$$\rho = \sqrt{\frac{K}{M}}$$

TABLE.

MOMENTS OF INERTIA OF SOME REGULAR HOMOGENEOUS SOLIDS.

Axis of rotation passing through center of mass.	Value of K.
Sphere of radius R and mass M	$\frac{2}{5}MR^2$
Cylinder of radius R and mass M, axis of cylinder is the axis rotation	$\frac{1}{2}MR^2$
Slim rod of length L and mass M, axis of rotation at right angles to rod	$\frac{1}{12}ML^2$
Rectangular parallelopiped of length L and breadth B, axis of rotation at right angles to L and B	$\frac{1}{12}M(L^2+B^2)$

Using the values of K in the above table, this equation shows that the radius of gyration of a sphere is $\sqrt{\frac{2}{5}}$ times the radius of the sphere, the radius of gyration of a cylinder rotating about its axis of figure is $\sqrt{\frac{1}{2}}$ times the radius of the cylinder, and the radius of gyration of a long, slim rod rotating about an axis at right angles to the rod and passing through its center of mass is $\sqrt{\frac{1}{12}}$ times the length of the rod.

If a rotating body be imagined to be divided into particles of equal mass, then the radius of gyration may be defined as the square-root-of-the-average-square of the distances of all the particles from the axis.

Angular momentum. The product of the moment of inertia of a rotating body and the angular velocity of the body is called the *angular momentum* or *spin momentum* of the body, that is to say, the spin momentum of a body is equal to $K\omega$, where K is the moment of inertia of the body about its axis of spin, and ω is its spin velocity in radians per second. The spin momentum of a body cannot change except by the action of an outside torque upon the body. This fact is known as *the principle of the conservation of angular momentum.* The following experiment furnishes a very striking illustration of this fact. A person stands upon a stool which turns freely about a vertical axis (on ball bearings), and he is set slowly rotating with his arms extended. By bringing his arms down to his sides he decreases his moment of inertia very considerably and produces a very great increase in his angular velocity. The change of moment of inertia is very much greater if the person holds weights in his hands. The fact here described is also illustrated by the formation of a whirlpool when a slowly rotating liquid flows out through a hole in the bottom of a bowl. The movement of the fluid towards the axis of the bowl reduces the moment of inertia of the system, and, according to the principle of constancy of angular momentum, the angular velocity is very greatly increased.*

63. Kinetic energy of a rotating body. A rotating wheel evidently stores kinetic energy because it can do work while being brought to rest. The kinetic energy of a rotating body is given by the equation

$$W = \tfrac{1}{2} K\omega^2, \tag{35}$$

in which everything is expressed in c.g.s. units. The proof of this equation gives, perhaps, a clearer idea of the significance of

*This matter is discussed in Art. 123.

moment of inertia than the discussion of equation (32) given in the foregoing article. Consider the rotating wheel shown in Fig. 54. The linear velocity of the particle Δm is $r\omega$, according to equation (29), and therefore the kinetic energy of this particle is

$$\Delta W = \tfrac{1}{2}\Delta m \cdot r^2\omega^2$$

according to equation (26).

Consider in this way all of the particles of the wheel and we have

$$\Delta W = \tfrac{1}{2}\Delta m \cdot r^2\omega^2$$
$$\Delta W_1 = \tfrac{1}{2}\Delta m_1 \cdot r_1^2\omega^2$$
$$\Delta W_2 = \tfrac{1}{2}\Delta m_2 \cdot r_2^2\omega^2$$
$$\Delta W_3 = \tfrac{1}{2}\Delta m_3 \cdot r_3^2\omega^2$$

whence, by adding, we have

$$W = \tfrac{1}{2}\omega^2\Sigma r^2 \cdot \Delta m$$

and by comparing this equation with equation (31) we have equation (35).

64. Relation between moments of inertia about parallel axes. Let K be the moment of inertia of a body of mass M about a

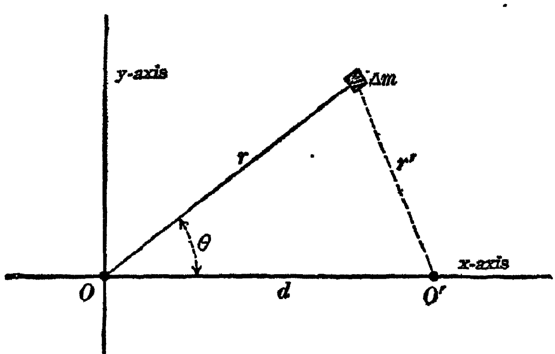


Fig. 56.

given axis passing through the center of mass of the body, and let K' be the moment of inertia of the body about another axis

parallel to the first and distant d from the center of mass, then

$$K' = K + d^2M \tag{36}$$

Let O, Fig. 56, be the center of mass of the body, chosen as the origin of co-ordinates, let K be the moment of inertia of the body about an axis through O perpendicular to the plane of the paper, and let K' be the moment of inertia of the body about an axis through O' also perpendicular to the plane of the paper. Consider a sample particle of the body Δm, distant r from O and distant r' from O', and of which the coordinates are x and y.

By trigonometry we have

$$r'^2 = r^2 + d^2 - 2rd \cos \theta \tag{i}$$

From equation (31) we have

$$K' = \Sigma r'^2 \Delta m \tag{ii}$$

whence, substituting the value of r' from equation (i) we have

$$K' = \Sigma r^2 \cdot \Delta m + \Sigma d^2 \Delta m - 2d\Sigma r \cos \theta \cdot \Delta m \tag{iii}$$

but $\Sigma r^2 . \Delta m$ is equal to K, and $\Sigma d^2 \Delta m$ is equal to d^2M. Furthermore $\Sigma r \cos \theta . \Delta m$ is equal to $\Sigma x . \Delta m$, which is equal to zero according to equation (22) since the origin of coordinates is chosen at the center of mass of the body. Therefore equation (iii) reduces to equation (36).

65. Equivalent mass of a rolling wheel. If a wheel and axle is set moving by an applied force F as shown in Fig. 57*a*, a string being wrapped around the axle and fastened to a rail as indicated

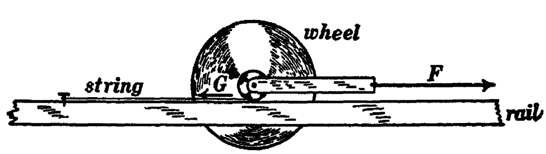


Fig. 57*a*.

in the figure, then a backward force G will be exerted on the wheel and axle by the string, and the net unbalanced force which is producing forward acceleration of the wheel will be $F - G$. The same conditions exist when a wheel is set rolling along a

track, that is to say, while the wheel is gaining forward velocity the track exerts a backward force at the rim of the wheel like the force G in Fig. 57*a*. Also, while a railway car is gaining forward velocity, the track exerts a backward force at the rim of each wheel like the force G in Fig. 57*a*. This is especially true in the case of a trolley car * which has rapidly rotating motor armatures geared to the car axles. Therefore, a given force applied to a car produces less forward acceleration than it would produce if the car were to slide along a frictionless track instead of moving forwards on rolling wheels.

When it is desired to take this effect into account in the discussion of the motion of a car in practice, it is usual to assign to the car a *fictitious mass* in excess of its actual mass so that the forward acceleration produced by an applied force F acting alone would be the same as the forward acceleration produced by the two forces F and G on the actual car. This fictitious mass is called the *equivalent mass* of the car and wheels.

It is sufficient for present purposes to discuss the equivalent mass of a rolling wheel by itself. The equivalent mass of the wheel is most easily defined as that mass M which would store the whole kinetic energy of the rolling wheel at a forward velocity equal to the linear velocity v of the wheel. Thus we may write

$$\tfrac{1}{2}Mv^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}K\omega^2 \tag{i}$$

in which m is the actual mass of the wheel, K is the moment of inertia of the wheel, and ω is the angular velocity of the wheel. The angular velocity ω, however, satisfies the eqaution

$$v = r\omega \tag{29bis}$$

and therefore, substituting v/r for ω in equation (i) we have

$$\tfrac{1}{2}Mv^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}\frac{K}{r^2}v^2$$

or

$$M = m + \frac{K}{r^2} \tag{37}$$

*This statement refers to an ordinary trolley car with idle motors, the car being hauled along by another car or coasting down hill.

Examples. (*a*) The rolling motion of the wheels of a railway train causes the train to behave, in so far as the relation between linear acceleration and draw-bar pull of locomotive are concerned, as if its mass were greater than its actual mass by the

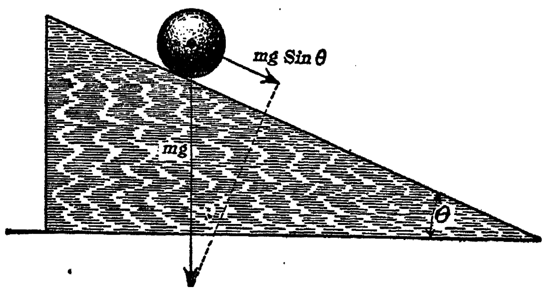


Fig. 57*b*.

amount nK/r^2, where n is the number of wheels, r is the diameter of the rolling circle of each wheel, and K is the moment of inertia of each wheel. In the case of an electric car with a geared motor, the moment of inertia of the motor armature can be reduced to an equivalent moment of inertia of wheel and thus be included in the value of K in equation (37), by multiplying the moment of inertia of the motor armature by the square of the gear ratio (ratio of the diameters of the rolling circles of the two gears).

(*b*) Consider a metal sphere of mass m and radius r rolling down an inclined plane as shown in Fig. 57*b*. The vertical pull

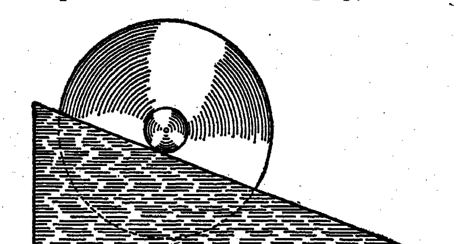


Fig. 58.

of the earth mg has a component parallel to the plane which is equal to $mg \sin \theta$, and this force would cause the ball to move with an acceleration equal to $g \sin \theta$ if the inclined plane were

frictionless so that the sphere would slide; but on account of the rolling motion the sphere behaves as if its mass were $\frac{7}{5}$ times m, according to equation (37), and therefore it rolls down the plane with an acceleration of only $\frac{5}{7}$ of $g \sin \theta$.

A wheel and axle rolling on a track as shown in Fig. 58 has a rolling circle of small radius r, so that its equivalent mass is very large, according to equation (37), and, therefore, such a wheel and axle rolls down an inclined plane with a very small acceleration.

66. Correspondence between translatory motion and rotatory motion. To every equation in translatory motion there is a corresponding equation in rotatory motion in which moment of inertia K takes the place of mass m, angle takes the place of distance, angular velocity ω takes the place of linear velocity v, angular acceleration α takes the place of linear acceleration a, and so on. The following table exhibits the pairs of corresponding equations.

TABLE.

Translatory motion.		Rotatory motion.	
$F=ma$	(3)	$T=K\alpha$	(32)
$W=Fd$	(24)	$W=T\phi$	(i)
$P=Fv$	(25)	$P=T\omega$	(ii)
$W=\frac{1}{2}mv^2$	(26)	$W=\frac{1}{2}K\omega^2$	(35)
$F=-kx$	(15)	$T=-\kappa\phi$	(38)
$k=4\pi^2n^2m^2$	(16)	$\kappa=4\pi^2n^2K$	(39)

Of these equations, those numbered (i), (ii), (38) and (39) have not been previously discussed; equation (i) refers to the work W done by the torque T in turning a body through an angle φ, axis of torque and axis of motion being parallel; and equation (ii) refers to the power P developed by a torque T which acts on a body rotating at angular velocity ω, axis of torque and axis of motion being parallel.

Equations (38) and (39) refer to harmonic rotatory motion, that is, to oscillatory motion about an axis, such as is exemplified by the motion of the balance wheel of a watch.

The equations of circular translatory motion correspond to the equations of the gyroscope to a limited extent as explained in Art. 72.

67. Rotatory harmonic motion. Consider a weight suspended by a steel wire. The weight will stand in equilibrium with the wire untwisted. If the weight is turned around the wire as an axis through the angle φ from this equilibrium position, then the twisted wire will exert a torque T on the weight tending to turn it back, and this torque will be proportional to φ, that is

$$T = -\kappa\varphi \tag{38}$$

in which the factor κ is a constant for a given wire; it is called the *constant of torsion* of the wire.

A weight suspended by a steel wire oscillates back and forth when the weight is turned about the wire as an axis and released, and the oscillatory motion of the weight constitutes what is called *harmonic rotatory motion.* Equation (38) is exactly similar in form to equation (15) (see table in Art. 66), and therefore the number n of oscillations of the weight per second and the moment of inertia K of the weight satisfy equation (39) which is exactly similar in form to equation (16). Therefore

$$\kappa = 4\pi^2 n^2 K \tag{39}$$

or, using $1/\tau$ for n, where τ is the period of one oscillation, we have

$$\kappa = \frac{4\pi^2 K}{\tau^2} \tag{40}$$

A weight hung by a wire and set oscillating about the wire as an axis, is called a *torsion pendulum.*

68. Use of the torsion pendulum for the comparison of moments of inertia. The constant of torsion, κ, equations (39) and (40), is nearly independent of the amount of weight supported by the wire, if the weight is not excessive, therefore, if two bodies are hung from the same wire, one at a time, and their respective periods of torsional vibration τ and τ' observed, then from equation (40) we have

$$\kappa = \frac{4\pi^2 K}{\tau^2} \tag{i}$$

and

$$\kappa = \frac{4\pi^2 K'}{\tau'^2} \tag{ii}$$

whence

$$\frac{K}{K'} = \frac{\tau^2}{\tau'^2} \tag{iii}$$

from which K' may be calculated if K is known. For example, one of the suspended bodies may be a homogeneous circular disk of which the moment of inertia is known (see table in Art. 62).

69. The gravity pendulum consists of a rigid body AB, Fig. 59, suspended so as to be free to turn about a horizontal axis O.

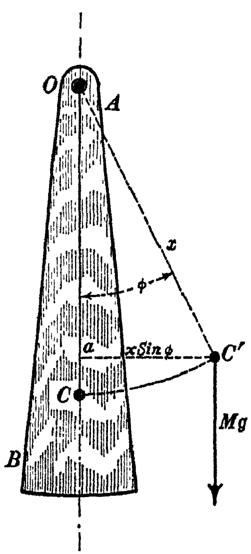


Fig. 59.

Let C, Fig. 59, be the center of mass of the body. This point C is vertically below O when the body is in equilibrium. Let the body be swung to one side through the angle φ, as shown. Then the force, Mg, with which the earth pulls the body tends to swing the body *back* to the vertical position with a torque T which is equal to the product of Mg and the length of the arm aC'. But the distance aC' is equal to x sin φ, where x is the distance OC. Therefore

$$T = Mgx \sin \varphi \tag{i}$$

When φ is small, then $\sin \varphi = \varphi$, very nearly, and equation (i) becomes

$$T = Mgx \cdot \varphi^* \tag{ii}$$

Comparing this with equations (38) and (40), we find that

$$\frac{4\pi^2 K}{\tau^2} = Mgx \tag{41}$$

*Of course this equation may be written $T = -Mgx \cdot \phi$ because T tends to reduce ϕ.

in which K is the moment of inertia of the body about the axis O, τ is the period of one complete pendulous vibration of the body, and g is the acceleration of gravity.

A pendulum such as here described is sometimes called a *physical pendulum* to avoid confusion with the ideal *simple pendulum* described in Art. 43.

The simple pendulum. An ideal pendulum consisting of a particle of mass M suspended by a weightless cord, or rod, of length l is called a *simple pendulum.* The moment of inertia of such a pendulum about the supporting axis O is $K = Ml^2$, according to equation (31). Furthermore, the center of mass of a simple pendulum is, of course, at the center of the suspended particle. Therefore, for the simple pendulum, we may write Ml^2 for K, and l for x in equation (41), whence we have

$$\frac{4\pi^2 l}{\tau^2} = g \tag{42}$$

or

$$l = \frac{\tau^2 g}{4\pi^2} \tag{43}$$

in which l is the length of a simple pendulum, τ is the period of one complete vibration of the pendulum and g is the acceleration of gravity.

Equivalent length of a physical pendulum. The length of a simple pendulum which would have the same period of vibration as a given physical pendulum is called the *equivalent length* of the given physical pendulum. Now, according to equation (43), the length of a simple pendulum, of which the period of one vibration would be τ, is $l = \tau^2 g/4\pi^2$. Therefore, solving equation (41) for $\tau^2 g/4\pi^2 (= l)$ we have

$$l = \frac{K}{Mx} \tag{44}$$

in which l is the equivalent length of a given physical pendulum, K is the moment of inertia of the pendulum about its axis of

support, M is the mass of the pendulum, and x is the distance from the point of support to the center of the mass of the pendulum.

The point in the line OC, Fig. 59, which is at a distance $l (= K/Mx)$ from O is called the *center of oscillation* of the pendulum. This point is also called the *center of percussion* of the pendulum for the reason that if the pendulum is started or stopped by a horizontal hammer-blow at this point no side force is produced on the supporting axis. See Art. 71.

70. The determination of gravity. The most accurate determination of the acceleration of gravity is made by means of the pendulum. This determination would be a very simple thing if it were feasible to construct a simple pendulum, in which case equation (42) could be used for calculating gravity from the measured length, l, of the simple pendulum and its observed period τ. The determination of the acceleration of gravity by means of an actual pendulum depends, however, upon the determination of the moment of inertia of the pendulum, as is evident from equation (41), and the moment of inertia of a body cannot be determined with great accuracy. This difficulty is obviated by means of the so-called *reversion pendulum* which was devised by Henry Kater in 1818.

Fig. 60.

A simple form of Kater's pendulum is shown in Fig, 60. A stiff metal bar has two knife-edges, from either of which it may be swung as a pendulum, and two weights, WW, which may be adjusted until the period τ of one vibration of the pendulum is the same whether it be swung from a or b. Then the distance between the knife-edges a and b is the equivalent length of the pendulum and may be used for l in equation (43).*

Comparison of the values of gravity at two places by means of the pendulum. If the same pendulum be swung at two places in

*See equation (46), below.

succession and its respective periods τ and τ' observed, we have from equation (41)

$$\frac{4\pi^2 K}{\tau^2} = Mgx \tag{i}$$

and

$$\frac{4\pi^2 K}{\tau'^2} = Mg'x \tag{ii}$$

in which g and g' are the respective values of the acceleration of gravity at the two places. Dividing equation (i) by equation (ii), member by member, we have

$$\frac{g}{g'} = \frac{\tau'^2}{\tau^2}$$

From this equation the value of g may be accurately determined at any place in terms of its known value at another place, by observing the values of τ and τ' of an ordinary pendulum, every precaution being taken to avoid variations of dimensions of the pendulum due to temperature or to careless handling. Most of the gravity determinations of the United States Coast and Geodetic Survey are made in this way, the value of g at Washington having been once for all determined with the greatest possible accuracy by means of Kater's pendulum.

The accompanying table gives the value of g in centimeters per second per second at several places as determined by the pendulum.

Theory of the reversion pendulum.—Consider a body of mass M, its center of mass at O, Fig. 61. Let O', O, and O'' be co-linear points; let τ' and τ'' be the vibration periods of the body swung as a pendulum from O' and O'' respectively; and let K, K', and K'' be the moments of inertia of the body about O, O', and O'' respectively. From equation (41) we have

$$\frac{4\pi^2 K'}{\tau'^2} = Mgx \tag{i}$$

and

$$\frac{4\pi^2 K''}{\tau''^2} = Mgy \tag{ii}$$

From equation (36) we have

$$K' = K + x^2M \tag{iii}$$

$$K'' = K + y^2M \tag{iv}$$

TABLE.

Locality.	Latitude.	Longitude.	Elevation.	Value of g (not Reduced to Sea-level).
Boston, Mass.	42° 21′ 33″	71° 03′ 50″	22 meters.	980.382
Philadelphia, Pa.	39 57 06	75 11 40	16 "	980.182
Washington, D.C. ..	38 53 20	77 01 32	10 "	980.100
Ithaca, N. Y.	42 27 04	76 29 00	247 "	980.286
Cleveland, O.	41 30 22	81 36 38	210 "	980.227
Cincinnati, O.	39 08 20	84 25 20	245 "	979.990
Terre Haute, Ind....	39 28 42	87 23 49	151 "	980.058
Chicago, Ill.	41 47 25	87 36 03	182 "	980.264
St. Louis, Mo.	38 38 03	90 12 13	154 "	979.987
Kansas City, Mo. ..	39 05 50	94 35 21	278 "	979.976
Denver, Col.	39 40 36	104 56 55	1638 "	979.595
San Francisco, Cal...	37 47 00	122 26 00	114 "	979.951
Greenwich	51 29 00	0 00 00	47 "	981.170
Paris	48 50 11	2 20 15	72 "	980.960
Berlin	52 30 16	13 23 44	35 "	981.240
Vienna	48 12 35	16 22 55	150 "	980.852
Rome	41 53 53	12 28 45	53 "	980.312
Hammerfest	70 40 00	22 38 00	——	982.580

Substituting these values of K' and K'' in (i) and (ii), we have

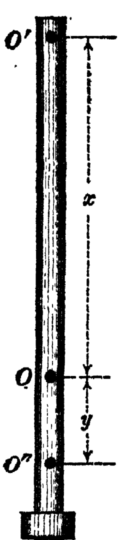


Fig. 61.

$$\frac{4\pi^2(K + x^2M)}{\tau'^2} = Mgx \qquad \text{(v)}$$

$$\frac{4\pi^2(K + y^2M)}{\tau''^2} = Mgy \qquad \text{(vi)}$$

Eliminating K/M from (v) and (vi), we have

$$\frac{4^2\pi(x^2 - y^2)}{x\tau'^2 - y\tau''^2} = g \qquad (45)$$

If $\tau' = \tau''$, we may cancel $(x-y)$, *provided* $(x-y)$ *is not equal to zero*, giving

$$\frac{4\pi^2(x + y)}{\tau'^2} = g \qquad (46)$$

(1) If the pendulum has been adjusted by repeated trial, so that $\tau' = \tau''$, then equation (46) enables the calculation of g, when $(x+y)$ and τ' have been observed.

(2) If the pendulum has not been adjusted, equation (45) enables the calculation of g, when x, y, τ', and τ'' have been observed.

(3) If the pendulum has been roughly adjusted, so that τ' and τ'' are nearly equal, then *equal and opposite errors in x and y very nearly annul each other in their influence upon the value of g as calculated by equation* (45). Therefore equation (45) gives g very accurately when τ' and τ'' are nearly equal, $(x+y)$ being measured with great accuracy, and

x measured roughly. The value of y is taken from $(x+y)-x$, so that its error may counteract the error due to the roughly measured value of x. The position of the center of mass O, Fig. 61, is found with sufficient accuracy for the rough measurement of x by balancing the pendulum horizontally on a knife edge.

Note. When $x=y$, equation (46) is not necessarily true, since it has been derived from equation (45) by cancelling $(x-y)$, which is zero.

71. Motion of a rigid body when struck with a hammer. When an unbalanced force continues to act upon a body for an appreciable length of time, the problem of determining the motion of the body is complicated by the fact that, as the body moves, the force generally changes its point of application, or its value, or its direction, or all three of these things may change simultaneously. The force due to a hammer blow, however, is of such short duration that the actual movement of the body during the time that the force acts is negligible, and the problem of finding the motion produced by the hammer blow is quite simple. A hammer blow is called an *impulse* and it is measured by the product of the average value, F, of the force exerted by the hammer and the short time t that the force continues to act. The impulse of a hammer blow when the hammer is brought to rest by the blow is equal to the momentum mv of the hammer. This is evident from the following considerations: Let F be the average value of the force acting to stop the hammer. Then $F = ma$, where m is the mass of the hammer and a is the average rate at which it loses velocity while stopping. Multiplying both members of this equation by the time which elapses during the stopping of the hammer, we have

$$Ft = mat$$

but the average rate a at which the hammer loses velocity multiplied by the time t is the total initial velocity of the hammer, and therefore the average force F exerted by the hammer while it is stopping multiplied by the time t occupied in stopping is equal to the momentum mv of the hammer. When m is expressed in grams and v in centimeters per second, then mv is expressed in dyne-seconds. When m is expressed in pounds

(mass) and v in feet per second then mv is expressed in poundal-seconds.

A rigid stick, AB, Fig. 62*a*, is struck with a hammer in the direction of the arrow, h, at a point distant x above the center of mass, O, of the stick. The motion imparted to the stick by the blow is a combination of translatory motion and rotatory motion, but the combination of a constant translatory motion and a constant rotatory motion is exactly the kind of motion which is performed by a rolling wheel, and therefore the hammer blow causes the stick to move as if the stick were attached to a weightless circular hoop, CC, and this hoop allowed to roll without friction on a straight rail. The center of the rolling circle, CC, is at the center of mass of the stick, and the radius, y, of the rolling circle depends upon the distance, x, and upon the

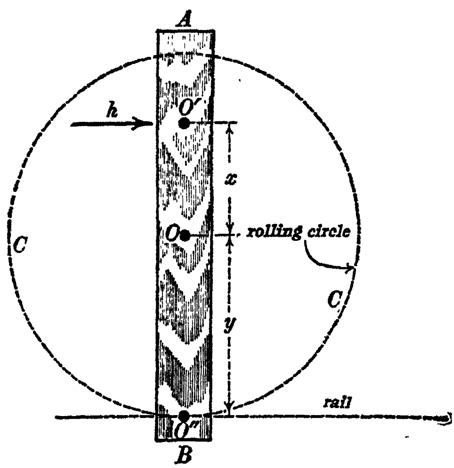


Fig. 62*a*.

ratio of the moment of inertia of the stick (about O) to the mass of the stick, according to equation (47). At the instant of the hammer blow the motion of the stick is equivalent to a simple motion of rotation about the point O''.

To analyze the effect of the hammer blow, the translatory motion and the rotatory motion may be treated separately. Re-

garding the translatory motion, we know from, Art. 48, that the velocity imparted to the center of mass is the same as if the whole mass of the stick were concentrated there and acted upon directly by the total force of the hammer. Let F be the average force due to the hammer, and t the time (very short) that it continues to act. Then F/M is the acceleration of the center of mass of the stick, and F/M multiplied by t is the velocity imparted to the center of mass, M being the mass of the stick.

As to the rotatory motion of the stick, it is evident that Fx is the torque about O due to the force of the hammer, so that Fx/K is the angular acceleration of the stick during the time t, and Fx/K multiplied by t is the angular velocity imparted to the stick by the hammer blow, K being the moment of inertia of the stick about O.

Now the whole stick is moving to the *right* at a velocity Ft/M on account of the translatory motion, and any point at a distance r below O is moving to the *left* at a linear velocity equal to r times the angular velocity, Ftx/K; therefore, for the point O'' which is for the moment stationary, we must have, writing y for r,

$$\frac{Ftxy}{K} = \frac{Ft}{M}$$

or

$$xy = \frac{K}{M} \tag{47}$$

which determines the radius y of the rolling circle when x and K/M are given.

The problem of the base-ball bat. At the instant that a base-ball bat strikes a ball, the motion of the bat is a simple motion of rotation about a certain point O'' Fig. 62*b*; and, if the distances x and y satisfy equation (47), then the effect of the impact of bat and ball is to reduce the angular velocity of the bat about the point O'' without moving the point O''. The point of a bat which must strike a ball so that the impact may have no tendency to change the position of the point about which the bat is rotating at the instant of impact, is called the *center of percussion* of the

bat. The position of the center of percussion depends of course upon the position of the point O'' about which the bat is rotating at the instant of impact.

72. Precessional rotatory motion. The foregoing articles refer to rotation about a fixed axis, or, as in the case of a rolling wheel, to rotation about an axis which performs translatory motion. The axis of a rotating body may, however, change its direction continuously. We shall discuss here only the comparatively simple case * in which a symmetrical body spins about its axis of symmetry while at the same time the axis of spin rotates uniformly. This rotation of the axis of spin is called

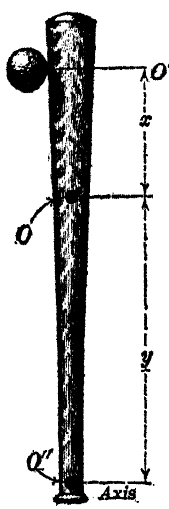


Fig. 62b.

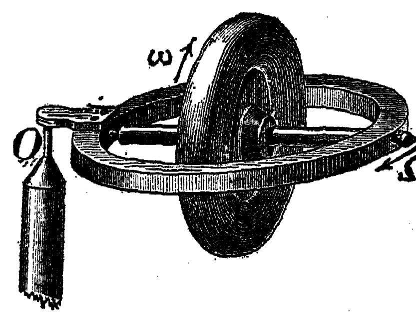


Fig. 63.

precession, and the axis about which the axis of spin rotates is called the *axis of precession*.

The *gyroscope* consists of a heavy wheel mounted on an axle which is pivoted in a metal supporting ring, as shown in Fig. 63.

*The student is referred to Poinsot's *Theorie Nouvelle de la Rotation des Corps*, which is perhaps the most intelligible account of the motion of a non-symmetrical rigid body. *Spinning Tops and Gyroscopic Motion* by H. Crabtree (Longmans, Green & Co., 1909) is perhaps the best simple treatise on this subject. The most complete treatise on the motion of a rigid body is *Advanced Rigid Dynamics* by E. J. Routh, Macmillan & Company, London, 1892.

The wheel is set in rapid rotation by wrapping a cord on the axle and giving the cord a vigorous pull. When the wheel is thus set rotating, the direction of the axle remains unaltered as long as no external twisting force, or torque, acts upon it; *an unbalanced torque is necessary to change the* **direction** *of the axis of a rotating body, just as an unbalanced force is required to change the* **direction** *of translatory motion of a particle.*

In order to describe precisely how an unbalanced torque changes the direction of the axis of a rotating body, it is very convenient to represent *angular velocity* and *torque* by lines in a diagram. To represent an angular velocity by a line, draw the line in the direction of the axis of spin and of such length as to represent to scale the value of the angular velocity in radians per second; to represent a torque by a line, draw the line in the direction of the axis of the torque and of such length as to represent to scale the value of the torque in dyne-centimeters. In each case an arrow-head is to be placed on that end of the line towards which a right-handed screw would travel if turned in the direction of rotation in the one case, or if turned in the direction of the torque in the other case.

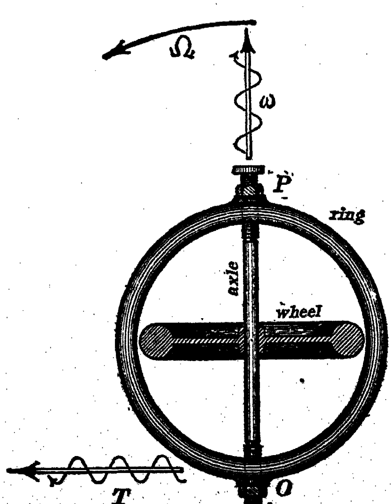


Fig. 64*a*.

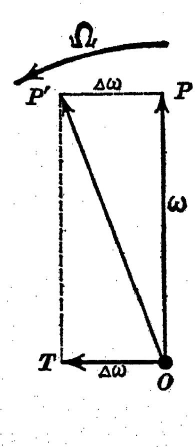


Fig. 64*b*.

Figure 64*a* is a top view of the gyroscope; the metal ring rests upon a supporting pivot underneath the ring at O, and the line OP, Fig. 64*b*, represents the angular velocity ω of the spinning wheel at a given instant. The pull of the earth on the wheel and ring produces an unbalanced torque about the axis OT, Fig. 64*a*. The effect of this unbalanced torque, during a short interval of time, is to impart to the wheel an additional angular velocity $\Delta\omega$ about the axis OT, and the resultant* angular velocity is then about the axis OP', Fig. 64*b*; that is, *the effect of the unbalanced torque T is to cause the axis of spin to sweep around O in the direction of the arrow Ω.*

This effect of an unbalanced torque upon a rapidly rotating body is also exemplified by the motion of a spinning top. Thus the line OP, Fig. 65, represents the angular velocity of a spinning top. The vertical pull of the earth, mg, produces an

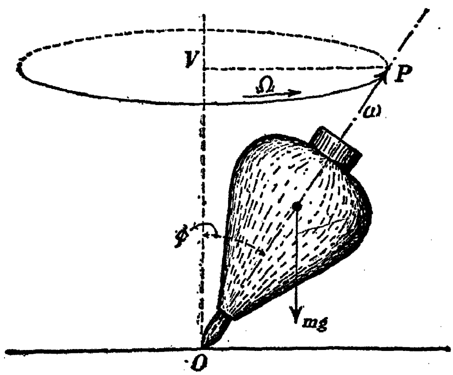


Fig. 65.

unbalanced torque about O, and the angular velocity produced by this unbalanced torque, by being added continuously to OP as a vector, causes the axis of spin OP to sweep round the vertical axis OV in the direction indicated by the arrow Ω.

The force required to constrain a particle to a circular orbit depends upon the mass of the particle and upon the linear acceleration which is involved in the continual change of direction of

*The vector addition of angular velocities is explained in Art. 79.

the velocity of the particle. The torque required to produce precession of a spinning body depends upon the moment of inertia of the body and upon the angular acceleration which is involved in the continual change of direction of the axis of spin. Precessional motion of a spinning body corresponds to translatory motion in a circle.

The torque required to produce precessional rotatory motion is given by the following equations:

$$T = \omega\Omega K \qquad \text{(i)*}$$

when the axis of spin is at right angles to the axis of precession as in Figs. 63 and 64, or

$$T = \omega\Omega K \sin\varphi \qquad \text{(ii)*}$$

when the axis of spin makes an angle φ with the axis of precession as in Fig. 65; in these equations ω is the angular velocity of spin in radians per second, Ω is the angular velocity of precession

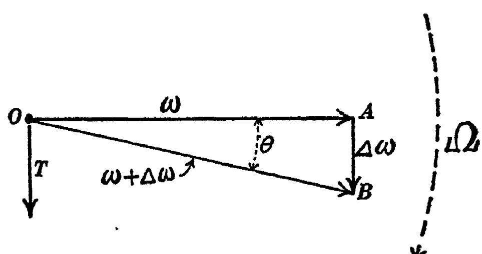


Fig. 66.

*These equations are rigorously correct only when the rotating body is symmetrical about the axis of spin and after steady precessional motion has been established, and equation (ii) is rigorously correct only when the rotating body is symmetrical about the point O in Fig. 65 as it would be if it were a rotating sphere with its center at O. Equation (ii) is approximately true in a case like that shown in Fig. 65 when the angular velocity of spin is very much greater than the angular velocity of precession. The precessional motion of the top in Fig. 65 about OV tends to increase the angle ϕ independently of the torque due to the force mg, and therefore the precessional motion of OP is more rapid than would be produced by the force mg alone if the top were symmetrical about the point O.

The student is referred to a very excellent treatise on *Spinning Tops* by Harold Crabtree, for a more complete discussion of this subject and especially for the discussion of the oscillations of a gyroscope or top (precessional motion not steady).

in radians per second and K is the moment of inertia of the body about the axis of spin.

Equation (i) may be established as follows: Let the line OA, Fig. 66, represent the angular velocity of spin of the body at a given instant, and let the dotted arrow Ω represent the angular velocity of precession. The angle θ turned by the axis of spin in a short interval of time Δt is $\theta = \Omega \cdot \Delta t$, and the increment of angular velocity which is represented by the line AB is equal to $\theta\omega$, θ being very small. Therefore

$$\Delta\omega = \omega\Omega \cdot \Delta t$$

whence

$$\frac{\Delta\omega}{\Delta t} = \omega\Omega$$

but $\Delta\omega/\Delta t$ is the angular acceleration of the rotating body, and the torque T which must act upon the spinning body to produce this angular acceleration is equal to the product of the angular acceleration and the moment of inertia of the body according to equation (32). Therefore we have

$$T = \omega\Omega K$$

Equation (ii) may be established in the same way by considering the component VP of the angular velocity of spin OP in Fig. 65.

The above analysis of the action of the gyroscope will hardly be convincing to the beginner on account of the fact that the action is analyzed in terms of the rather complicated and unfamiliar ideas, angular velocity, angular acceleration, moment of inertia and torque; it is, therefore, desirable to analyze the action of the gyroscope in terms of the fundamental ideas of linear velocity and acceleration, mass, and linear force. The analysis of the action of the gyroscope in terms of linear velocity and acceleration depends upon a relation which is sometimes called Coriolis' law. Given a straight tube AB, Fig. 67, which is rotating about the axis C at angular velocity Ω as indicated in the figure. In this tube is a ball m which is moving away from C at velocity v (if the ball were moving towards C its velocity v would be considered as negative). Under these assumed conditions *the sidewise acceleration, a, of the ball m is equal to $2\Omega v$,* that is

$$a = 2\Omega v \tag{v}$$

To derive this relation, the sidewise acceleration a may be considered in two parts. In the first place we have the acceleration which is associated with the continual

change of direction of the radial velocity v of the ball. This acceleration is equal to Ωv as shown in Fig. 68, and as explained in Arts. 18 and 38. In the second place, as the ball gets farther and farther away from the axis C, its actual sidewise velocity, due to the rotation of the tube, increases, but this sidewise velocity is equal to Ωr, according to equation (29), and therefore, since v is the rate at which r is changing it is evident that Ωv is the rate at which the sidewise velocity, Ωr is changing, as

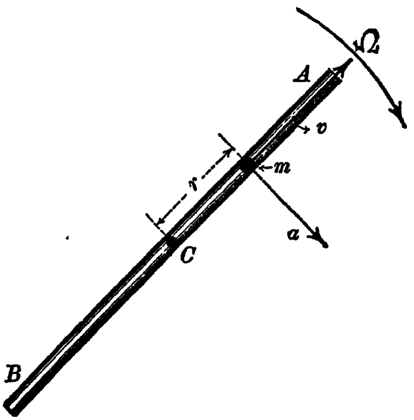


Fig. 67.

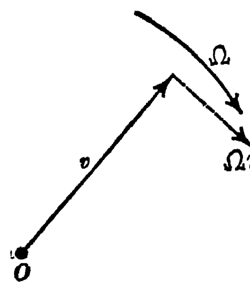


Fig. 68.

explained in Art. 17. The relation $a = 2\Omega v$ is used in the discussion of motion of steam engine governors, where the governor balls have a motion of rotation combined with a motion towards or away from the axis.

Consider now a circular disk AB, Fig. 69, spinning at angular velocity ω about its axis of figure O, and let the axis O be turning about CD at angular velocity Ω. Consider a sample particle m of the disk at a distance r from O as shown in Fig. 69. The velocity of m is $r\omega$, and the component of this velocity which is away from the axis CD is $r\omega \sin\theta$; and, therefore, the precessional rotation about the line CD involves an acceleration of m towards the reader which, according to equation (v), is

$$a = 2\omega\Omega r \sin\theta \qquad \text{(vi)}$$

It may be easily seen that this acceleration is *towards the reader* in quadrants 1 and 2, and *away from the reader* in quadrants 3 and 4, and, therefore, the forces required to produce these accelerations constitute a torque about the axis EF as indicated by the arrow T.

73. Examples of precessional rotation.* (*a*) *The precession of the earth's axis.* The attraction of the sun keeps the earth in

*A description of various practical aspects of gyrostatic action is given in an article by W. S. Franklin in the *Popular Science Monthly* of July, 1909. This article contains in particular a description and explanation of the Brennan gyrostatic mechanism for maintaining the equilibrium of a monorail car, and the article also includes a description of the gyrostatic action of the boomerang.

its orbit. The force of attraction of the sun upon the bulging equatorial portion *a*, Fig. 70, is *more* than sufficient to constrain this portion of the earth to its circular orbit around the sun, and

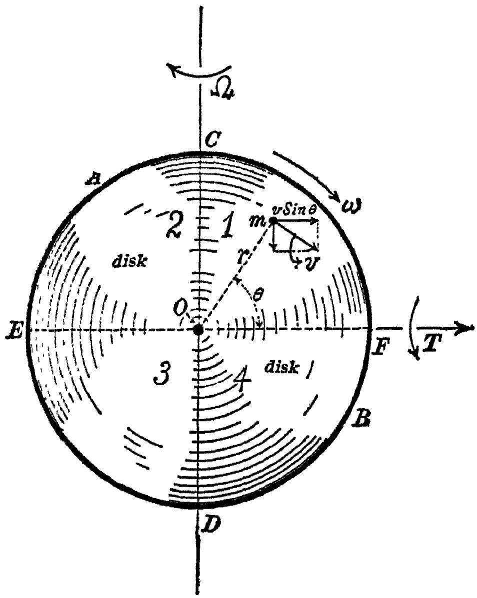


Fig. 69.

the force of attraction of the sun on the bulging equatorial portion *b* is less than sufficient to constrain that portion of the earth to

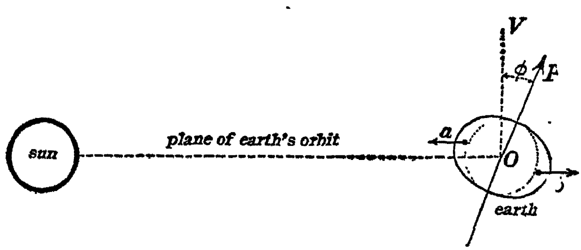


Fig. 70.

its circular orbit around the sun. Therefore, the earth is acted upon by an unbalanced torque about *O* which causes the earth's axis to describe a cone about the line *O V* which is at right angles

to the plane of the earth's orbit. The action of the moon is here ignored for the sake of simplicity.

(*b*) *A coin rolling along the floor* is, of course, rotating, and the instant the coin begins to be inclined to either side, the unbalanced torque due to gravity causes a precessional movement of the axis of the coin, and the coin describes a curved path in consequence of this precession.

(*c*) *Rotating parts of machines on ship-board.* The pitching and rolling of a vessel at sea causes, at each instant, a certain angular velocity Ω of the axis of a rotating machine part, and an unbalanced torque is immediately brought into existence. For example, when a steamer turns round, the propeller and propeller shaft change direction continuously; when a steamer rolls, the axis of a dynamo armature which is athwart ship changes its direction periodically; when a steamer pitches, the axis of the propeller and propeller shaft changes its direction periodically. In the case of a steamer driven by steam turbines the propeller shaft turns at high speed and the rotating member of the turbine is quite heavy, so that the pitching motion of such a vessel produces excessively large forces at the bearings which support the shaft. A turbine torpedo-boat of the British Navy went down in a heavy sea in 1899 or 1900, being probably broken in two by the very great forces produced by the pitching of the boat, and the consequent angular motion of the propeller shaft, forces which, perhaps, were not duly considered in the designing of the hull and supporting structure of the shaft.

(*d*) *The gyrostatic action of an automobile engine.* Figure 71*a* represents a view of an automobile as seen from above, *WW* being the fly-wheel of the engine with its axis or shaft *ab*. The automobile is represented as turning to the right as indicated by the curved arrow Ω. During a short interval of time the angular velocity of spin turns from the direction OA to the direction OB in Fig. 71*b*, and the line AB represents what is to be considered as the increment of angular velocity of spin during the interval, so that a torque T must act on the engine shaft as the

automobile turns to the right. To exert the torque T as represented in Fig. 71*b*, a force must push upwards on the front end *a* of the engine shaft and a force must push downwards on the rear end *b* of the engine shaft, or, in other words, the front end *a* of the engine shaft must push downwards on its bearing and the

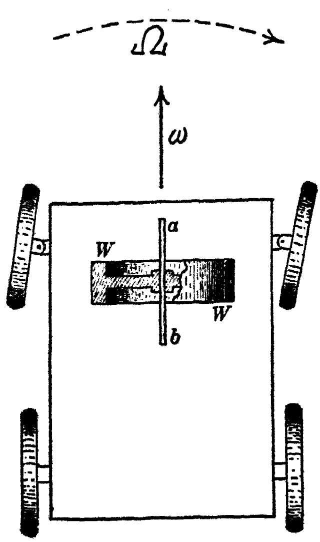


Fig. 71*a*.

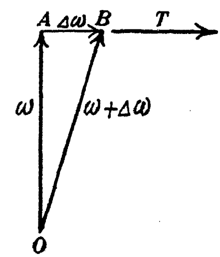


Fig. 71*b*.

rear end *b* of the engine shaft must pull upwards on its bearing. This reaction of the engine shaft, while the automobile is turning as shown in Fig. 71*a*, tends to push the front wheels of the automobile against the ground with excessive force, and to lift the rear wheels off the ground.

(*e*) *The use of the gyroscope (or gyrostat as it is sometimes called), for preventing the rolling of a ship at sea.* A rapidly rotating wheel is hung from a hinge so that its axis may swing back and forth in a vertical plane, *a vertical plane including the keel of the boat*, as shown in Fig. 72*a*. The lower end of the axis is attached to a rod which connects to a piston in what is called a dash-pot. When the vessel rolls about the keel as an axis, the axis of the gyrostat oscillates back and forth, and the effect of the friction of the piston in the dash-pot is the same as if the rolling of the

ship were hindered by excessive friction, and thereby the motion of rolling is greatly reduced. A small German torpedo boat, 115 feet long by 12 feet beam, was recently equipped with a gyro-

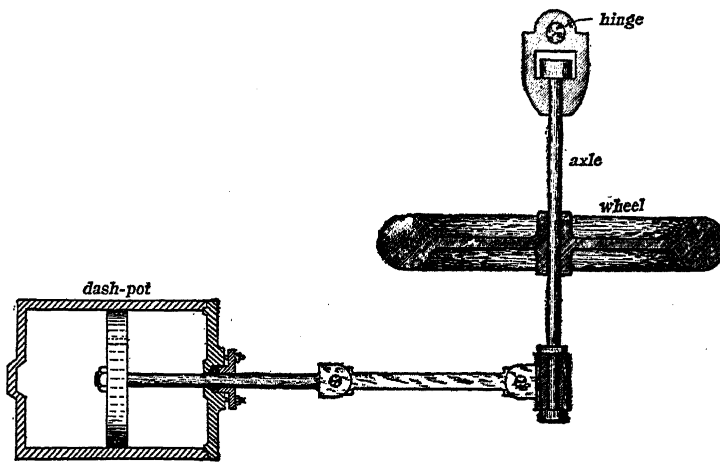


Fig. 72*a*.

stat arranged as shown in Fig. 72*a*.* The gyrostat wheel was 3.3 feet in diameter, it had a mass of 1,100 pounds and it was

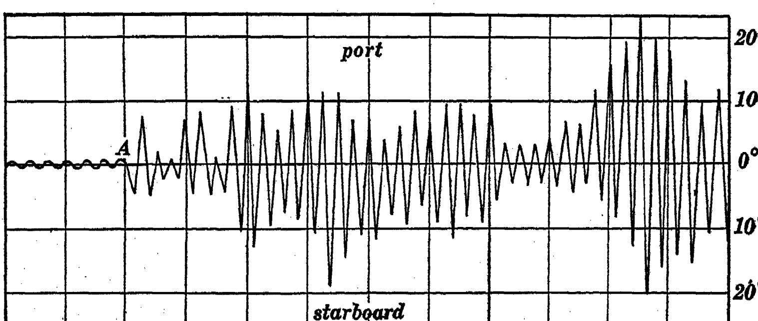


Fig. 72*b*.

driven at a speed of 1,600 revolutions per minute. The effect of this arrangement is shown in Fig. 72*b*, in which the ordinates measured from the line marked 0° represent angular amplitudes

*See paper by Otto Schlick, translated in *Scientific American Supplement*, for January 26, 1907.

of rolling oscillations. To the right of the point A the curve shows the rolling when the gyrostat is inoperative, and to the left of the point A the curve shows the rolling when the gyrostat is in action.

KINEMATICS OF A RIGID BODY.*

74. Motion of a rigid body in a plane.—A rigid body is said to move in a plane when all points of the body which lie in the plane remain in it. For example, a rotating wheel moves in a plane, the connecting rod of a steam engine moves in a

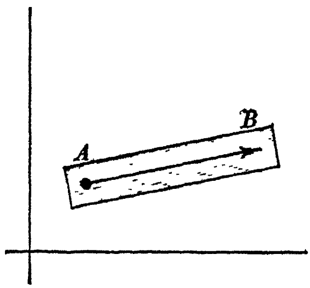


Fig. 73.

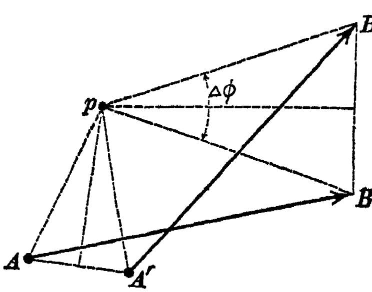


Fig. 74.

plane. Consider a rigid body AB, Fig. 73, moving in the plane of the paper. The position of the body is completely indicated by the position of the line AB fixed in the body. This line is called the *index line.*

After any change in the position of a rigid body moving in a plane, a certain line in the body, perpendicular to the plane, is in its initial position, and the given displacement is equivalent to a rotation about that line as an axis. Let AB and $A'B'$, Fig. 74, be the positions of the index line before and after the displacement. Join AA' and BB'. Erect perpendiculars from the middle points of AA' and BB' intersecting at p. From the similarity of the triangles pAB and $pA'B'$ it is evident that the same part of the body is at p before and after the displacement, and that the line through p perpendicular to the paper is the line about which the body may, by simple rotation, move from its initial to its final position. The angle $\Delta\phi$ of this rotation is the angle subtended by AA' or BB' as seen from p.

75. The instantaneous motion of a rigid body moving in a plane in any manner is a motion of rotation about a definite line called the *instantaneous axis* of the motion. Let the displacement, shown in Fig. 74, be that which takes place in a

*The discussion of the dynamics of a rigid body should properly be preceded by a discussion of the kinematics of a rigid body. This, however, has not been done because most of the discussion of the dynamics of a rigid body can be based upon the simple idea of rotation about a fixed axis. Thus the rotatory motion of a rolling wheel is in no way different from what it should be if the translatory motion did not exist.

short interval of time Δt; then $\Delta\phi/\Delta t$ is the instantaneous angular velocity of the body, and the line through p, perpendicular to the paper, is the instantaneous axis. During a finite interval of time the motion of a body may be irregular, but the motion of a body during an interval of time approaches uniformity as that interval approaches zero. Therefore the motion of a body during a short interval of time is the simplest motion which can produce the actual displacement which occurs during the interval.

76. Composition of angular and linear displacements. Consider an angular displacement $\Delta\phi$ of a body about the point p, Fig. 75, bringing the point O to O'; and the linear displacement Δl parallel and equal to $O'O$, bringing O' back to O. These two displacements are together equivalent to an angular displacement $\Delta\phi$ about O, bringing Op to Op'. Let the distance of p from the line OO' be r; then, if $\Delta\phi$ is small, $\Delta l = r\Delta\phi$.

77. Resolution of motion in a plane. From Arts. 75 and 76 it follows that the instantaneous motion of a rigid body in a plane may be resolved into a motion of rotation about an arbitrary point combined with a certain linear velocity. Consider the actual displacement represented in Fig. 75, namely, a rotation about O bringing p to p'. This displacement is equivalent to an equal angular displacement $\Delta\phi$, about the arbitrary point p, together with the linear displacement $O'O$ or pp'. Let this linear displacement be Δx, and let Δt be the interval which elapses during the displacement. The actual angular velocity $\Delta\phi/\Delta t$ about the point O (the instantaneous center) is equivalent to an angular velocity $\Delta\phi/\Delta t$ about the point p combined with a linear velocity $\Delta x/\Delta t$ parallel to pp'.

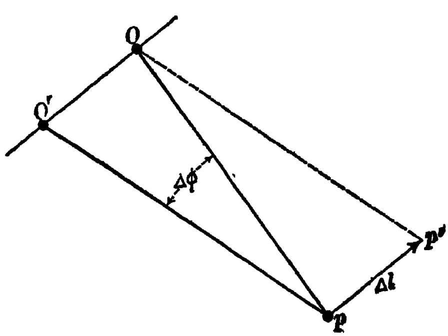


Fig. 75.

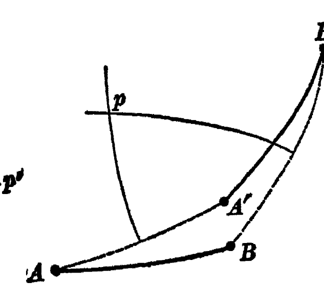


Fig. 76.

78. Motion of a rigid body with one point fixed. If a rigid body, one point of which is fixed, is displaced in any manner whatever, a certain line in the body will be in its initial position after the displacement, and the given displacement will be equivalent to a rotary movement about this line as an axis.

Proof. Consider a spherical shell of a body having its center at a fixed point. Let AB, Fig. 76, be an arc of a great circle on this spherical shell; the position of AB fixes the position of the body, and AB is called the *index line*. Let the movement of the body bring AB to $A'B'$. Connect AA' and BB' by arcs of great

circles. Draw great circles bisecting AA' and BB' at right angles. The point p at the intersection of these circles bisecting AA' and BB' has the same position relative to AB and $A'B'$, so that this point of the shell is in its initial position, and the line drawn from the center of the spherical shell to the point p is the axis about which the given movement can be produced by rotation.

The instantaneous motion of a rigid body about a fixed point is a motion of simple rotation at definite angular velocity about a definite line called the instantaneous axis of the motion.

79. Vector additions of angular velocities. Consider an angular velocity about the axis a, Fig. 77, and another angular velocity about the axis b; the two angular

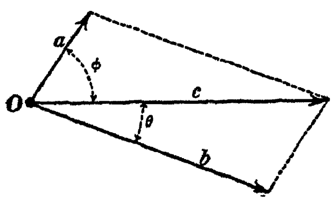


Fig. 77.

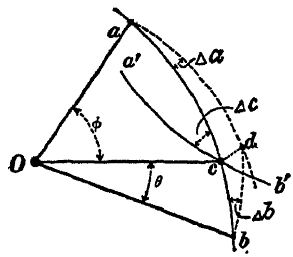


Fig. 78.

velocities are together equivalent to an angular velocity about the axis c, the respective angular velocities being proportional to the lengths of the lines a, b and c.

Outline of proof. Imagine a sphere constructed with its center at O, Fig. 77, and let a and b, Fig. 78, be the points where the lines a and b, Fig. 77, cut the sphere. Imagine a very small rotation Δa about Oa followed by a very small rotation Δb about Ob, bringing the great circle ab to the position $a'b'$. The point of

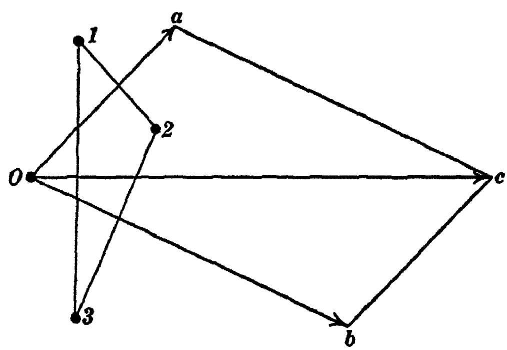


Fig. 79.

intersection of ab and $a'b'$ is the point where the resultant axis Oc cuts the sphere, and the angle Δc is the amount of rotation about Oc which is equivalent to the two rotations Δa and Δb; then it can be shown* that the three angles Δa, Δb and

*This matter is discussed in detail in *Spinning Tops* by Harold Crabtree (Longmans, Green, & Co., 1909) pages 34–35.

Δc are related to each other as the lengths of the three lines a, b and c, Fig. 77, and that the two arcs ac and cb subtend angles which are equal to ϕ and θ respectively of Fig. 77.

80. **Vector addition of torques.** Let the lines a and b, Fig. 79, represent two given torques and let it be required to show that a and b are together equivalent to the torque c. Draw the lines 1–2, 2–3, and 1–3 perpendicular to and bisected by a, b and c respectively. The lengths of these lines are proportional to the lengths a, b and c. Imagine the torque a to be due to a unit upward force at 1 and a unit *downward* force at 2 (upward and downward being perpendicular to the plane of the paper), then the torque b is equivalent to a unit of *upward* force at 2 and a unit of downward force at 3; but the upward force and downward force at 2 annul each other, so that we have left only a unit of upward force at 1 and a unit of downward force at 3, which constitute a torque about the line c proportional to the length of c.

PROBLEMS.

95. A body starts from rest and after 10 seconds it is rotating 55 revolutions per second. What is the average angular acceleration? Express the result in radians per second per second. Ans. 34.6 radians per second per second.

96. In what terms is moment of inertia expressed: (*a*) When length is expressed in centimeters and mass in grams? (*b*) When length is expressed in inches and mass in pounds? (*c*) When length is expressed in feet and mass in pounds? The unit moment of inertia in case (*a*) is the c. g. s. unit. How many c. g. s. units of moment of inertia are there in the unit involving the inch and the pound, and in the unit involving the foot and the pound? Ans. (*a*) grams $\times$ centimeters squared; (*b*) pounds $\times$ inches squared; (*c*) pounds $\times$ feet squared. One pound-inch^2 = 2,926 gram-centimeters^2; one pound-foot^2 = 421,300 gram-centimeters^2.

97. Calculate the moment of inertia of a uniform slim rod, length 3.1 feet ($= l$) and mass 3.6 pounds ($= m$), about an axis passing through the center of the rod and at right angles to the length of the rod.

(*a*) Calculate K from the formula $K = ml^2/12$.

(*b*) Calculate K approximately by multiplying the mass of each 0.1 foot of the rod by the square of its estimated mean distance from the center of the rod.

(*c*) Calculate the radius of gyration of the rod.

Ans. (*a*) K = 2.883 pounds × feet squared; (*b*) 2.879 pounds × feet squared; (*c*) radius of gyration 0.895 feet.

98. Calculate the moment of inertia of a circular disk, radius 1.7 feet, mass 4.25 pounds, about the axis of figure.

(*a*) Calculate K from the formula given in the table in Art. 62. The circular disk is of course a very short cylinder.

(*b*) Calculate the radius of gyration of the disk. Ans. K = 6.14 pounds × feet squared; radius of gyration of disk is 1.22 feet.

99. (*a*) Calculate the moment of inertia of the rod, problem 97, about an axis passing through the end of the rod and perpendicular to the rod.

(*b*) Calculate the moment of inertia of the disk, problem 98, about an axis passing through the edge of the disk parallel to the axis of figure of the disk. Ans. (*a*) 11.52 pounds × feet squared. (*b*) 18.42 pounds × feet squared.

100. A circular disk 5 feet diameter, weighing 1,200 pounds is mounted upon a shaft 6 inches in diameter. The disk, set rotating at 500 revolutions per minute and left to itself, comes to rest in 75 seconds. Calculate average (negative) angular acceleration while stopping, calculate average torque acting to stop the disk, and calculate the frictional force at the circumference of the shaft. Ans. 0.698 radians per second per second; torque acting to stop the disk is 81.7 pound-feet; frictional force 327 pounds-weight.

101. What is the kinetic energy of the disk specified in problem 100 when the speed is 500 revolutions per minute? Ans. 160,640 foot-pounds.

102. A metal disk 12 inches in diameter and weighing 25 pounds, has a cylindrical hub projecting on each side. Each hub is 1 inch in diameter and weighs ¼ of a pound (total mass 25.5 pounds). What is the moment of inertia of the whole?

The hubs of this disk roll on a track which drops 1 inch vertically in each foot of horizontal distance; find how fast the disk

gains linear velocity in rolling down this track. Ans. 3.1254 pounds $\times$ feet squared; 0.037 feet per second per second.

103. A slim rod 2 feet long and having a mass of 2.5 pounds is suspended by a wire. The wire is attached to the middle of the rod and the rod hangs in a horizontal position. The rod, set vibrating about the wire as an axis, makes 50 complete vibrations in 10 minutes 25 seconds. What torque would be required to twist the wire through one complete turn? Ans. 0.0414 pound-feet.

104. An irregular body is suspended by the same wire that is specified in problem 103, and, set vibrating about the wire as an axis, it makes 37 complete vibrations in 10 minutes. What is its moment of inertia? Ans. 1.404 pounds $\times$ feet squared.

105. A uniform slim rod 4 feet long is hung as a gravity pendulum at a point distant 6 inches from the end of the bar. Calculate its equivalent length as a pendulum. Ans. 2.39 feet.

106. A pendulum clock rated in Boston and carefully transported to Hammerfest would gain how many seconds per day? Ans. 97 seconds gained per day.

See table in Art. 70.

107. The connecting rod of a steam engine weighs 19.5 pounds. Its center of mass is distant 18 inches from the center of the hole which fits the crank pin, and when it is swung as a gravity pendulum about the point *a*, Fig. 107*p*, it makes 100 complete

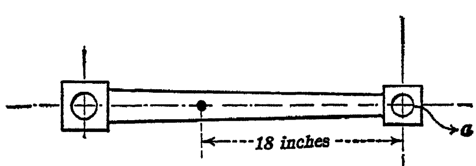


Fig. 107*p*.

vibrations in 2 minutes and 50.2 seconds. The diameter of the hole at *a* is 1¾ inches. What is the moment of inertia of the connecting rod about its center of mass? Ans. 24.2 pounds $\times$ feet^2.

Note. The moment of inertia of the connecting rod about point a may be calculated from the equation of the pendulum. The moment of inertia about the center of mass may then be calculated as explained in Art. 64.

108. A slim stick 5 feet long and having a mass of 10 pounds is held in a vertical position and struck a horizontal blow with a hammer at a point 18 inches from the upper end, the upper end being released at the instant of the blow. The impulse of the blow is 80 pounds-weight-seconds. Find the translatory velocity imparted to the stick (velocity of its center of mass), find the angular velocity about its center of mass, and find the position of the point in the stick which remains stationary for a moment after the hammer blow. Ans. 257.6 feet per second; 123.6 radians per second; the point is 2.083 feet below the middle of the stick.

Note. To solve this problem it is best to reduce the impulse of 80 pounds-weight-seconds to poundal-seconds by multiplying by 32.2. Then the translatory velocity of the stick is found by dividing the impulse in poundal-seconds by the mass of the stick in pounds. Multiplying the impulse in poundal-seconds by the distance between the center of mass of the stick and the point where the hammer strikes gives the *torque impulse* in poundal-foot-seconds, and dividing this torque impulse by the moment of inertia of the stick gives the angular velocity in radians per second.

109. Find the distance from the axis of suspension of the slim rod described in problem 105, to the point where the rod may be struck horizontally with a hammer without causing a side force to be exerted on the axis of suspension. Compare this distance with the "equivalent length" of the rod as a gravity pendulum. Ans. 2.39 feet; it is equal to the "equivalent length" of the rod as a gravity pendulum.

110. A water wheel is connected to its belt-pulley by a shaft. Find the torque, in pound-feet and in pound-inches, tending to twist the shaft when the water wheel develops 200 horse-power at a speed of 600 revolutions per minute. Ans. 1,750 pound-feet, or 21,000 pound-inches.

Note. The two equations $W = T\phi$ and $P = T\omega$ correspond exactly to equations (24) and (25) as explained in Art. 66. The only difficulty involved in the use of the equations $W = T\phi$ and $P = T\omega$ is to keep the units straight, as it were.

111. An electric motor, running at 900 revolutions per minute, develops 15 horse-power. Find the torque with which the field magnet acts upon the rotating armature, neglecting friction. Express the result in pound-feet and in pound-inches. Ans. 87.6 pound-feet, or 1,051 pound-inches.

112. The armature shaft of a ship's dynamo is athwart ship, and the armature is driven clockwise as seen from the port side of the vessel. Describe accurately the forces with which the bearings act upon the armature shaft as the vessel rolls. Specify the directions of these forces when the port side of the vessel is rising, and when the the port side of the vessel is falling.

This problem refers to the forces which arise from the rotatory motion of the armature. The port side of a vessel is on the left hand of a person facing the bow. Ans. When the port side of the ship is rising, the precessional torque exerted by the bearings upon the armature shaft is such that its reactions tend to turn the ship to port; the torque is reversed when the port side of the ship is falling.

113. A side-wheel steamboat is suddenly turned to port, and the gyrostatic action of the paddle wheels causes the boat to list. In which direction does the boat list, to starboard or port? Why? Ans. To starboard.

114. The vessel described in problem 113 is steered in a circle 150 feet in radius at a velocity of 25 feet per second, and the vessel lists 5° because of the gyrostatic action of the paddle wheel and shaft. To produce a 5° list when the boat is standing still requires a weight of 10 tons to be shifted from the center of the boat to a point 15 feet from the center. The paddle wheels make 75 revolutions per minute. Find the moment of inertia of the axle and wheels. Ans. 7,362,000 pound-feet^2.

115. A locomotive rounds a curve of radius 528 feet at a speed of 30 miles per hour. The diameter of the driving wheels is 6 feet and each pair of drivers and the connecting axle has a moment of inertia of 37,000 pound-feet^2. Find the torque acting on each pair of drivers due to the precession. How does this precession

modify the force with which the wheels push on the two rails? Ans. 1,413 pound-feet; it decreases the pressure of the wheels on the inner rail and increases the pressure on the outer rail.

116. A torpedo boat makes a complete turn in 84 seconds and its propeller rotates at a speed of 270 revolutions per minute. The moment of inertia of the propeller is 2,000 pound-feet.^{2} Required the precessional torque on the propeller shaft. In what direction does this torque tend to bend the shaft? Ans. 132.2 pound-feet. This torque will be in such a direction as to depress the stern and raise the bow of the boat if the propeller is rotating in a clockwise direction as viewed from behind and if the boat is turning to port.

117. A high speed engine with its shaft athwart ship, makes 240 revolutions per minute. The rim of the fly-wheel has a radius of 3 feet and a mass of 600 pounds. Calculate the moment of inertia of the wheel (rim). The maximum angular velocity attained by the vessel in rolling is $\frac{1}{10}$ radian per second. Calculate the maximum torque acting on the fly-wheel shaft due to gyrostatic action. Ans. 5,400 pound-feet^2; 424.1 pound-feet.

CHAPTER VII.

ELASTICITY (STATICS).

81. Stress and strain. When external forces act upon a body and tend to change its shape, the body is distorted more or less, and the external forces are balanced by the tendency of the distorted body to return to its original shape. The distortion of a body always brings forces into action between the contiguous parts of the body throughout. These force actions between contiguous parts of a distorted body are called *internal stresses;* and the total reaction of the distorted body, which balances the external distorting force, is called the *integral stress* of the body.

The actual movement of the point of application of an external force which distorts a body is called the *integral strain* of the body, and the change of shape of each small part of the distorted body is called the *internal strain.* Thus the elongation of a wire under tension, the shortening of a column under compression, the angular movement of the end of a rod under torsion, the depression of the middle of a beam which is loaded at its center, and the decrease of volume of a body which is subjected to hydrostatic pressure are integral strains; and the total stretching force acting on the wire, the total load on the column, the total torque tending to twist the rod, the total load at the middle of the beam, and the hydrostatic pressure which acts on a body, are integral stresses. In each of these cases, furthermore, each small part of the body is distorted, and force actions exist between contiguous parts of the body throughout. These are called the internal strains and the internal stresses respectively.

82. Homogeneous and non-homogeneous stresses and strains. It is generally the case in a distorted body, that each small part of the body is differently distorted, and that the internal stress varies from point to point in the body. For example, the dif-

ferent parts of a bent beam, or of a twisted rod, are differently distorted, and the internal stress varies from point to point; the pressure of the atmosphere decreases and the air becomes less and less dense with increasing altitude above the level of the sea; the pressure at a point in a body of water increases with the depth beneath the surface, and the water is more and more compressed as the pressure increases; the stress in a long cable, which is suspended in a mine shaft, increases from the lower end upwards, and the extent to which each portion of the cable is stretched increases with the stress.

When the force action between contiguous parts of a body is the same at every point in the body, the stress is said to be

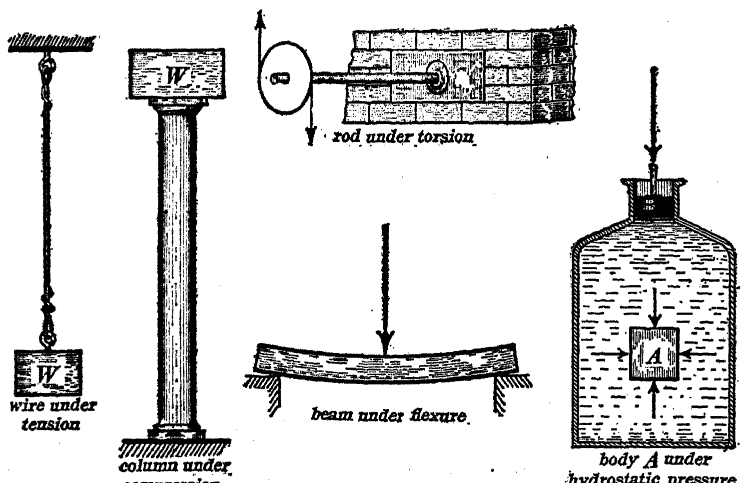


Fig. 80.

homogeneous; and when every part of a body is similarly distorted, the strain is said to be *homogeneous*. Thus each part of a rod under tension or compression is similarly distorted as shown in Figs. 86*a* and 86*b*, and the force action or stress is the same at every point as shown in Figs. 85*a* and 85*b*; the steam in a boiler is under the same pressure throughout (gravity negligible), and the degree of compression of every portion of the steam is the same; the water in the high pressure cylinder of a hydraulic

press is under the same pressure throughout (gravity negligible), and the degree of compression of every portion of the water is the same.

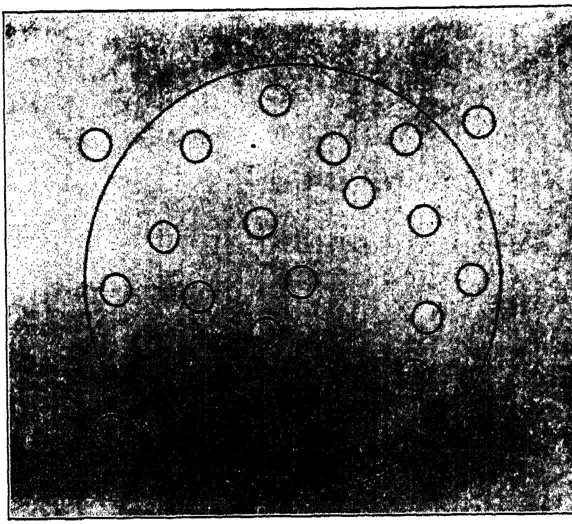


Fig. 81.

The distinction between homogeneous and non-homogeneous strains is shown in Figs. 81, 82, and 83, which are photographs

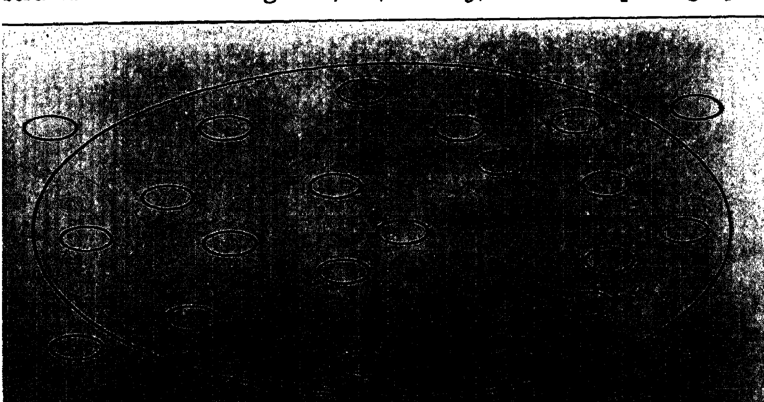


Fig. 82.

of a thin rubber sheet upon which a large circle and a number of small circles were drawn. Figure 81 shows the unstrained sheet,

Fig. 82 shows the sheet homogeneously strained, and Fig. 83 shows the sheet non-homogeneously strained. The small circles are changed to ellipses in Fig. 82 and in Fig. 83, but in Fig. 82 the ellipses are all alike and their axes are in the same direction, whereas, in Fig. 83, some of the ellipses are more elongated than others and their axes are not parallel. In a homogeneous strain a *large* portion of a substance is distorted in a manner exactly similar to the distortion of each *small* portion of the substance. This is shown in Fig. 82 in which the large ellipse is exactly the same shape as the small ones. In a non-homogeneous strain a *large* portion of a substance is irregularly distorted. This is

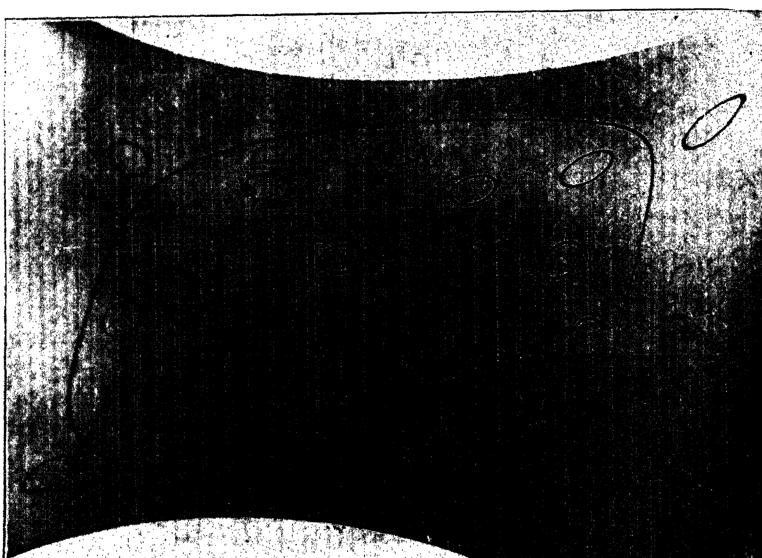


Fig. 83.

shown in Fig. 83 by the irregular curve into which the large circle has been converted by the strain. *It is a fact of fundamental importance in the theory of elasticity, that, however irregularly a body may be distorted, any small portion of the body suffers that simple kind of distortion which changes a sphere into an ellipsoid, or which, in the case of a thin sheet of rubber, changes a circle into an ellipse.* That is, the change of shape of any small portion of

a distorted body consists of an increase or decrease of linear dimensions in three mutually perpendicular directions, and, in some cases, this simple kind of distortion is accompanied by a slight rotation of the small parts of the body. Thus, in Fig. 90, which represents a portion of a bent beam, the short straight lines were all horizontal or vertical in the unbent beam.

The effect of a sharp groove in a body which is under stress is a matter of very great practical importance. The effect is, in general, to produce an excessive concentration of stress in the material at the bottom of the groove, and a crack or fracture is almost sure to develop, unless the material is plastic so that the bottom of the groove is broadened by yielding. Consider, for example, a beam in which a sharp groove is cut, as shown in Fig. 84. The fine lines in this figure represent the lines of stress

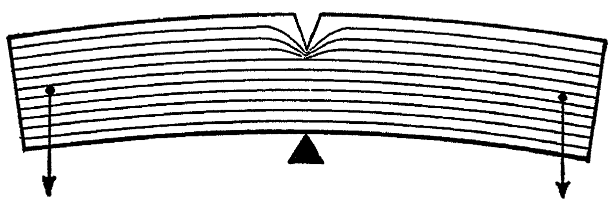


Fig. 84.

when the beam is bent, and the crowding together of these lines as they pass under the groove represents the concentration of stress above referred to.

The most striking illustration of the effect of a sharp groove in a body under stress is furnished by a piece of glass in which there is a minute crack. A piece of glass without a crack will stand a very considerable stress, but if the stress "flows" round the end of a crack, the stress is concentrated and the crack extends indefinitely. A pane of window glass or a glass tumbler is worthless when a crack once starts. A less familiar illustration of the effect of a sharp groove is furnished by the method commonly employed for breaking a bar of steel; thus a steel rail, which normally withstands the tremendous stresses due to the weight of a locomotive, can be broken in two by a hammer

blow if a nick is made across the top of the rail with a sharp chisel. Sharp re-entrant angles are always carefully avoided in the designing of those parts of structures which are intended to sustain stress.

83. Solids and fluids. Everyone is familiar, in a general way, with the three classes of substances, solids, liquids, and gases. A *solid* can withstand, for an indefinite length of time, a stress which tends to change its shape. A solid which recovers from distortion (strain) when stress ceases to act, is said to be *elastic*. Thus good spring steel recovers almost completely from a moderate amount of distortion when the distorting force (stress) ceases to act. A solid which does not recover from strain when the stress ceases to act, is said to be *plastic*. Thus lead and wax are plastic solids. No solid, perhaps, recovers completely from distortion; and, on the other hand, every plastic substance, perhaps, is slightly elastic. Thus the best spring steel does not completely recover from even a slight distortion, and when the distortion is great the steel takes a very decided permanent set; and even wax is slightly elastic, as is shown by the distinct metallic ring of a large cake of beeswax or paraffine when it is struck with a hammer.

A *fluid* is a substance which, at rest, cannot sustain a stress which tends to change its shape. While a fluid is actually changing shape, however, it does sustain a stress which tends to change its shape. Thus a stream of syrup falling from a vessel is under tension like a stretched rope, and the effect of this tension is to continually lengthen each portion of the stream of syrup as it falls. A fluid at rest always pushes normally against every portion of a surface which is exposed to the action of the fluid. Thus the steam in a boiler pushes outwards on the boiler shell at each point, the water in a vessel pushes normally against the walls of the vessel at each point, and the atmosphere pushes normally against every portion of an exposed surface. A fluid in motion, however, may not push normally against an exposed surface. Thus, a water pipe is subject only to a bursting force

if the water is at rest; but if the water flows through the pipe, it has a slight tendency to drag the pipe along with it.*

A *liquid* is a fluid, like water or oil, which can have a free surface, such as the surface of water in a glass. A *gas*, on the other hand, is a fluid which completely fills any containing vessel.

84. Hooke's law. Elastic limit. Robert Hooke discovered, in 1676, that what we have called the integral strain of a body is quite accurately proportional to what we have called the integral stress.† Thus, the elongation of a wire under tension is proportional to the stretching force, the shortening of a loaded column is proportional to the load, the angular movement of the end of a rod under torsion is proportional to the torque which acts on the rod, and the depression of a loaded beam is proportional to the load.

Elastic limit. Hooke's law is quite accurately true for distinctly elastic substances like steel, but it does not apply to plastic substances, and even for elastic substances like steel there is a limit, called the *elastic limit*, beyond which stress and strain are no longer even approximately proportional. When an elastic substance is strained beyond its elastic limit it does not return to its original size or shape when the stress ceases to act, but takes what is called a *permanent set.* Liquids and gases, however, return to their exact initial volume when relieved from pressure, provided the temperature has not changed, that is, liquids and gases may be said to be perfectly elastic, but when a liquid or gas is compressed the diminution of volume (integral strain) is not proportional to the increase of pressure (integral stress) except when the increase of pressure is fairly small. This is at once evident in the case of gases when we consider that they conform to Boyle's law as explained in Art. 101.

*A jet of water issuing from the end of a pipe pushes backwards on the pipe, as every fireman knows. This backward force is due to the normal force with which the water pushes on the inner walls of the pipe where the pipe bends.

†It follows from this experimental fact that the strain at each point of a distorted elastic body is proportional to the stress at that point.

85. Limitations and plan of this chapter. The phenomena which are associated with the distortion of bodies are excessively complicated. Let one consider the swaying of objects in the wind, the bending and compression of structures under load and their vibration with sudden variations of load; let one think of all the familiar properties of brittle substances like chalk and glass, of plastic substances like clay and wax and of elastic substances like steel and rubber; let one consider that all of the phenomena of sound are due to the vibrations of bodies, and to wave movements in the air, and, in many cases, to wave movements in water and in solids, all of which have to do with distortion and compression; and let one think that compression and expansion and local changes of shape are involved inextricably in nearly every case of flow of air and water. Let one think of all of these things and then consider whether it is not necessary to bring the mind to a narrow view before any clear line of argument can be pursued relative thereto!

Of all the great variety of solid substances, having almost every imaginable degree of elasticity, plasticity, hardness, and brittleness, and ranging in strength from sun dried clay to the toughest steel, we are chiefly concerned with the behavior under stress of those which are used as materials of construction; and, in addition, we are here concerned with the tendency of increase of pressure to reduce the volumes of liquids and gases.

A substance, like wood, which has a grained structure, is said to be *aeolotropic* (pronounced ē′-o-lo-trop′-ic). Most crystalline substances, and rolled and drawn metal are aeolotropic. A substance, like glass or water, which does not have a grained structure, is said to be *isotropic*. The behavior under stress of aeolotropic substances is very complicated; these complications need not be considered, however, for practical purposes, because substances having a grained or fibrous structure are generally subjected to stresses parallel to the grain, as in beams, and ropes and wires. The difference between aeolotropic substances and isotropic substance is ignored in this chapter

Types of stress and strain. In the discussion of the behavior of bodies under stress, it is necessary to consider three simple types of stress and strain. Thus we have *longitudinal stress* and *longitudinal strain*, which is the type of stress and strain in a rod under tension or in a column under compression; we have *hydrostatic pressure* and *isotropic* strain*, which is the type of stress and strain in a body subjected to hydrostatic pressure; and we have *shearing stress* and *shearing strain*, which is the type of stress and strain which exists (non-homogeneously) in a twisted rod. A discussion of the first two types of stress and strain is sufficient for most practical purposes, and, therefore, the discussion of shearing stress and shearing strain is given in small type preceding the outline of the general theory of stress and strain.

Treatment of non-homogeneous stresses and strains. In the following discussion, the behavior of a substance under each type of homogeneous stress and strain is first considered, and the ideas so developed are used as a basis for the discussion of important cases of non-homogeneous stresses and strains. For example, the discussion of the bent beam is based upon the discussion of homogeneous longitudinal stress and strain, and the discussion of the twisted rod is based upon the discussion of homogeneous shearing stress and strain.

LONGITUDINAL STRESS AND STRAIN.

86. Longitudinal stress. Figure 85*a* represents a portion of a rod under tension. Let F be the total force tending to stretch the rod, and let A be the sectional area of the rod; then the stretching force per unit of sectional area is $F/A(=P)$, and the force action between contiguous portions of the rod is as follows: Imagine a horizontal unit of area q anywhere in the material of the rod, the material on the two sides of q exerts a pull P across q, as shown in Fig. 85*a*; imagine a vertical unit of area q' drawn anywhere in the material of the rod, the material on the two sides

*When an isotropic substance, such as glass, is subjected to a hydrostatic pressure the substance is reduced in volume without being changed in shape. Such a strain is called an isotropic strain, for want of a better name.

of q' does not exert any force at all across q'. The force acting across any horizontal area of a units is, of course, equal to Pa. The force per unit area, P, is the *measure** of the longitudinal stress, and the direction of P is called the *axis of the stress*.

A rod under tension may be considered as a case of *positive* longitudinal stress, and a rod or column under compression may be considered as a case of *negative* longitudinal stress.

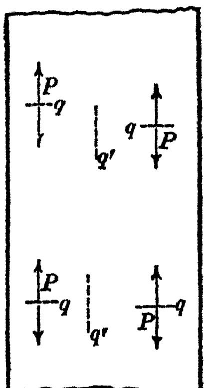


Fig. 85*a*.

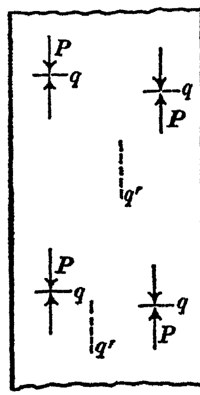


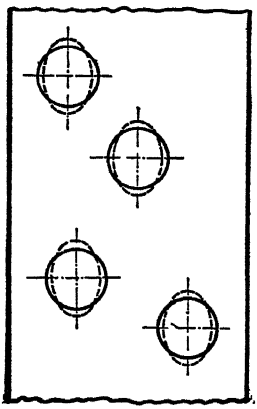
Fig. 85*b*.

87. Longitudinal strain. A rod under tension is longer than when it is not under tension, and, since each unit portion of the rod must be equally stretched, it is evident that the increase of length of the rod is proportional to its total initial length. Therefore it is most convenient to express the increase of length as a fraction of the total initial length; thus, a stretch of 2 thousandths means an increase of length equal to 2 thousandths of the total initial length of a rod. Let L be the initial length of a rod and let l be its increase of length under tension, then $l/L(=\beta)$ is the increase of length expressed as a fraction of the initial length, and, of course, the ratio of $L+l$ to L is equal to $1+\beta$. The fraction $\beta(=l/L)$ is used as the *measure* of the longitudinal strain.

*That is, the number which is used to specify the value of the stress. See Art. 9.

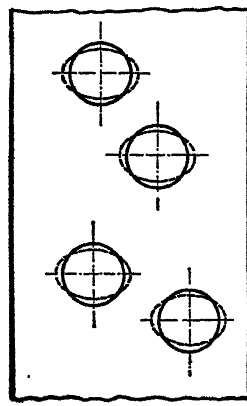
The character of the distortion of the parts of a rod under tension is shown in Fig. 86*a*, any spherical portion of the material of the rod becomes an ellipsoid of revolution. The major axis of the ellipsoid is $1 + \beta$ times the diameter of the original sphere.

Lateral contraction of a stretched rod. Poisson's ratio. When a rod is stretched, it contracts laterally as indicated by the dotted



rod under tension.

Fig. 86*a*.



rod under compression

Fig. 86*b*.

ellipses in Fig. 86*a*. This lateral contraction of a stretched body is very strikingly shown by a stretched rubber band. The lateral contraction d of a stretched rod is most conveniently expressed as a fraction of the original diameter D of the rod, and the fraction d/D we may represent by the letter β'. The elongation of a rod per unit length (β) bears, for a given substance, a fixed ratio to the lateral contraction per unit of diameter (β'), and this ratio is called *Poisson's ratio;* β is approximately four times as large as β' for steel, brass, and copper.

88. Stretch modulus of a substance. The elongation of a rod under tension is proportional to the stretching force (Hooke's law). Therefore the stretching force divided by the elongation is a constant for a given rod, and if we divide *stretching force*

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per unit area, P, *by elongation per unit of original length* $\beta(=l/L)$, the result is a constant which depends upon the material of the rod, but which is independent of the size and length of the rod. This constant, which is represented by the letter E, is called the *stretch modulus** of the material of which the rod is made. That is

$$E = \frac{P}{\beta} \tag{48}$$

The stretch modulus may be defined in a slightly different way as the factor which, multiplied by the elongation per unit length, gives the stretching force per unit sectional area of a rod under tension ($E\beta = P$); and it is evident from this definition that E is expressed in units of force per unit of area, inasmuch as β is a ratio of two lengths (l/L); in fact E is the force per unit sectional area which *would* double the length of a rod if the elongation would continue to be proportional to the stretching force. This, of course, is not true for elongations of more than a few parts per thousand.

89. Determination of stretch modulus. The stretch modulus may be determined by applying a known stretching force F to a rod of known length L and known sectional area A, and observing the increase of length l. Then $P = F/A$, and $\beta = l/L$, so that $E = P/\beta = FL/Al$. An easier method for determining

TABLE.

Values of the stretch modulus of various substances.

(In pounds-weight per square inch.)

Copper (drawn)	17,700,000
Steel (rolled)	29,800,000
Wrought iron	29,600,000
Cast iron	16,000,000
Glass	9,600,000
Oak wood	1,450,000
Poplar wood	750,000

the stretch modulus of a substance is by observing the deflection of a loaded beam as explained in Art. 91.

*Often called Young's modulus, or "the modulus of elasticity" by engineers.

90. Potential energy of longitudinal strain. The potential energy per unit volume of a stretched (or compressed) rod is equal to one half the product of the stress P and the strain β, or it is equal to one half of the product of the stretch modulus E and the square of the strain β. That is

$$W = \tfrac{1}{2} P\beta \tag{49}$$

or

$$W = \tfrac{1}{2} E\beta^2 \tag{50}$$

in which W is the potential energy per unit of volume of a stretched (or compressed) rod, P is the stretching (or compressing) force per unit sectional area of the rod; β is the increase (or decrease) of length per unit of original length, and E is the stretch modulus of the material. If P and E are expressed in pounds-weight per square inch, W is expressed in inch-pounds of energy per cubic inch of material.

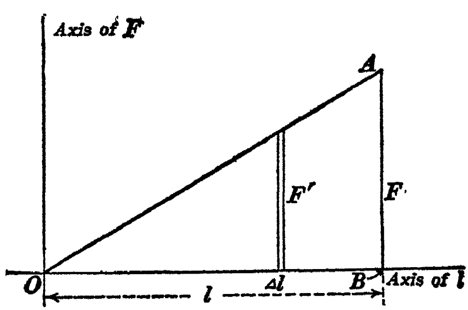


Fig. 87.

Proof. The increase of length l of a rod is proportional to the stretching force F, so that by plotting corresponding values of l and F we have a straight line OA, as shown in Fig. 87.

Imagine the stretching force to increase slowly from zero to F, the stretch at the same time increasing from zero to l. Let F' be an intermediate value of F, and let Δl be a very small increase of l due to a slight increase of F', then $F' \cdot \Delta l$ is the work done on the rod during the very slight increment of stretch Δl. But $F' \cdot \Delta l$ is the area of the narrow parallelogram shown in Fig. 87,

and therefore the total work done on the rod while the stretching force increases from zero to F is the total area OAB Fig. 87, which is equal to $\frac{1}{2}Fl$, so that the work done per unit volume of the rod is $\frac{1}{2}Fl$ divided by the volume AL of the rod. That is

$$W = \frac{1}{2}\frac{lF}{AL} = \frac{1}{2}P\beta$$

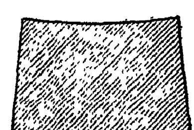
This proof may be stated in a slightly different way thus: The *average* value of the stretching force between zero stretch and the given stretch l, is $\frac{1}{2}F$, which, multiplied by the elongation l, gives the work done on the rod.

Equation (50) is derived from equation (49) by substituting $E\beta$ for P according to equation (48).

91. Discussion of a bent beam.* The simplest case of a bent beam is that which is shown in Fig. 88 in which a long beam is laid horizontally across two supports SS, and bent by hanging weights on the projecting ends as shown. The bending moment or torque of each weight is equal to Wx, and this bending torque acts on every portion of the beam between the supports SS, so that this portion of the beam becomes an arc of a circle.† All of

*When a beam is bent the stretched filaments on one side of the beam contract laterally and the compressed filaments on the other side of the beam expand laterally and the section of a beam originally square becomes distorted somewhat as shown in Fig. 89. This is very clearly shown by bending a rectangular bar of rubber, a lead

Section of beam before bending.



Section of beam after bending (exaggerated).

Fig. 89.

pencil eraser for example. This distortion of the section of a bent beam is usually very slight and it is neglected in the above discussion.

†To show that the bending moment to which the beam is subjected has the same value everywhere between the supports SS in Fig. 88, consider the portion of the beam $aaa'a'$. This portion of the beam is in equilibrium. The downward force W and the equal upward force at the support S constitutes a pure torque of which the value is Wx, and therefore the force with which the portion is acted upon at $a'a'$ by the remainder of the beam, that is the force action across the section $a'a'$, is a pure torque which is equal and opposite to Wx.

the filaments in the upper part of the beam are elongated, all of the filaments in the lower part of the beam are shortened, and certain filaments pp, which lie in what is called the *median* line or surface of the beam, remain unchanged in length. The beam

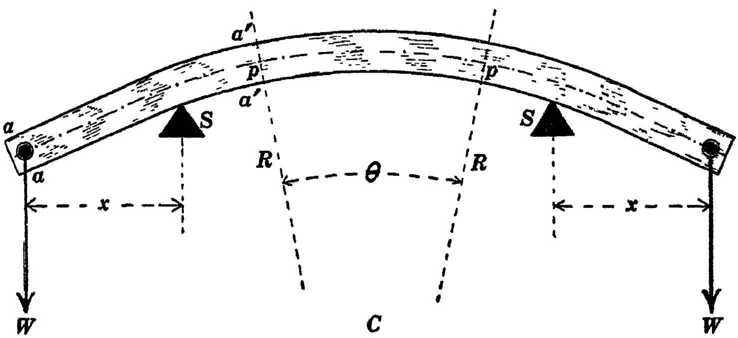


Fig. 88.

is everywhere under longitudinal strain as shown in Fig. 90; and the force action between contiguous parts of the beam is as shown in Fig. 91. These figures may be understood by comparing them

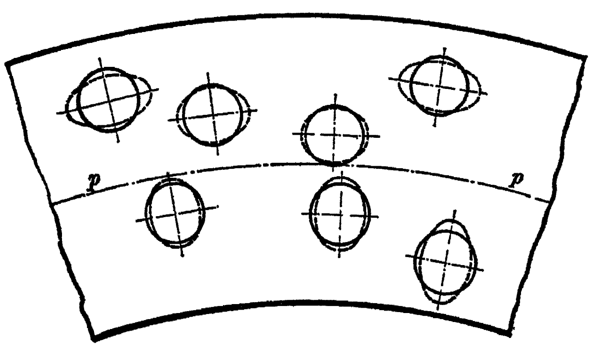


Fig. 90.

with Figs. 85 and 86. To find the value of the stress and strain at each point in the bent beam proceed as follows:

Consider the portion of the beam which lies between the radii R and R, Fig. 88. This portion is shown to a larger scale in Fig. 92. Let R be the radius of curvature of the median line pp, then the length of pp is equal to $R\theta$ *which is the original length of*

every filament of the beam between the radii R and R, Fig. 88. Consider a filament of the beam at a distance y above the median line,

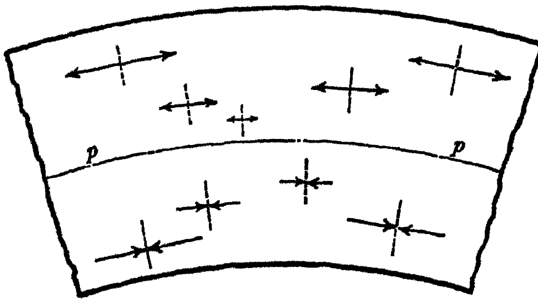


Fig. 91.

y being considered negative for filaments below the median line. The radius of curvature of this filament is $R + y$ and its length is $(R + y)\theta$. Therefore, the increase of length of this filament due to the bending of the beam is $y\theta$, and, expressing this increase of length as a fraction of the original length $R\theta$, we have

$$\beta = \frac{y}{R} \tag{i}$$

which expresses the value of the longitudinal strain at any point

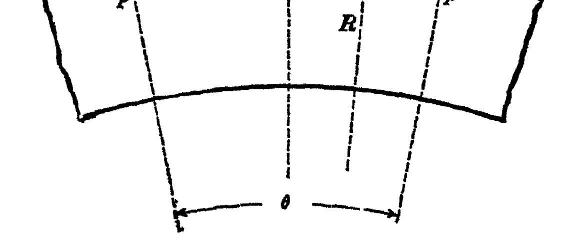


Fig. 92.

in the bent beam, β being positive where y is positive and negative where y is negative.

The longitudinal stress P (force per unit of area, as shown in Fig. 91) at each point of the beam is equal to $E\beta$, according to equation (48), where E is the stretch modulus of the material of the beam. Therefore, using y/R for β, we have

$$P = \frac{E}{R}\cdot y \tag{ii}$$

The total force action across a complete section ab of the beam shown in Fig. 93, that is, the total force action of the portion AA upon the portion BB, is a torque about an axis O

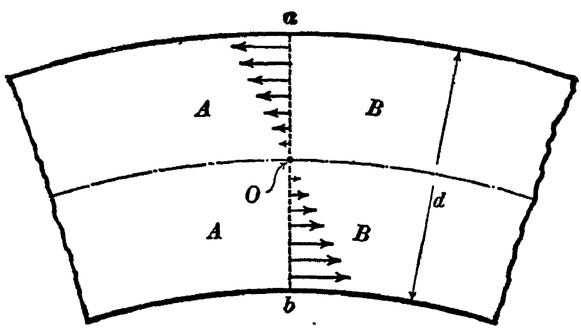


Fig. 93.

perpendicular to the plane of the figure. This axis is shown as the line OO in Fig. 94, which is a sectional view of the beam,

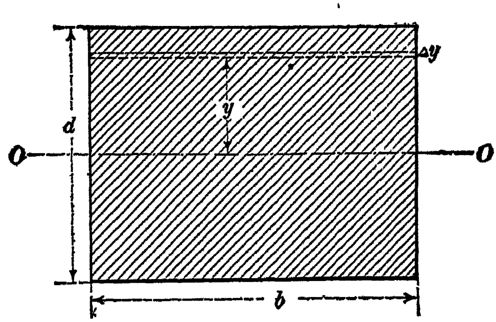


Fig. 94.

b being the breadth of the beam and d its depth. The torque action is expressed by the equation

$$T = \frac{1}{12} \cdot \frac{bd^3E}{R} \tag{iii}$$

But the torque action across every section of the beam between the two supports SS in Fig. 88 is equal to Wx, so that

$$Wx = \frac{1}{12} \cdot \frac{bd^3E}{R} \tag{iv}$$

from which E may be calculated when W, x, b, d and R are known.

Derivation of equation (iii). The portion of the beam between the two parallel lines Δy, in Fig. 94, has a sectional area equal to $b \cdot \Delta y$, and the force action across each unit of area of this portion of the beam is Ey/R, according to equation (ii) above. Therefore the total force action across the area $b \cdot \Delta y$ is $Eby \cdot \Delta y/R$, which, multiplied by the lever arm y, gives the torque action about OO which is due to the portion of the beam $b \cdot \Delta y$. Therefore,

$$\Delta T = \frac{Eb}{R} \cdot y^2 \cdot \Delta y$$

whence by integrating between the limits $y = -d/2$ and $y = +d/2$, we have equation (iii).

92. Important practical relations between longitudinal stress and strain.* The important practical aspects of the relation between longitudinal stress and strain may best be brought out by considering the behavior of a steel rod (a test piece) which is subjected to a continually increasing longitudinal stress. Thus the ordinates of the curve A in Fig. 95 represent the values of an increasing longitudinal stress in pounds-weight per square inch, and the abscissas represent the corresponding elongations, in hundredths, of a rod of ordinary bridge steel. *The curve B is the left-hand part of Curve A with abscissas magnified one hundred times.*

Up to a point p, the position of which is not very sharply defined, the strain (elongation per unit initial length) is very exactly proportional to the stress (stretching force per unit of sectional area); that is the stress-strain curve is a straight line from the origin to p. Beyond the point p the stress-strain line is very slightly curved until the point q is reached where the steel begins

*The student is referred to the splendid treatise on "The Materials of Construction" by J. B. Johnson, Wiley and Sons, 1898, for a full discussion of this subject.

to yield very greatly. This yielding takes place rather irregularly until the whole test-piece has yielded, it alters the temper of the steel, and the steel then sustains an increased stress which reaches a maximum at the point *t*. The metal then begins to be weakened by the continued increase of length, and finally the rod breaks at the point *b*.

The point *p* marks the *true elastic limit;* the point *q*, which is sometimes called the *yield point*, marks what is for practical

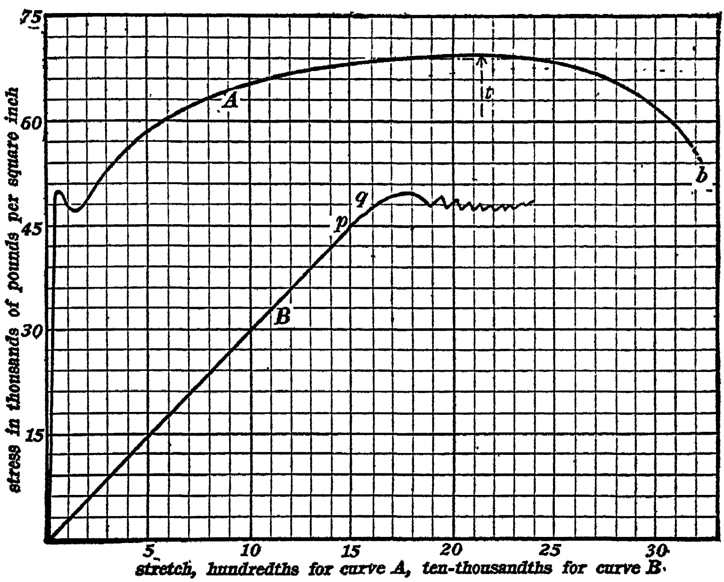


Fig. 95.

purposes* called the *elastic limit*, and the point *t* marks what is called the *tensile strength* of the steel. The most important of these points is the yield point or elastic limit, commerically so called, because the steel breaks after a very few applications of a stess that exceeds the elastic limit. Everyone is familiar with this fact in that a wire may be easily broken by bending it repeatedly beyond the elastic limit, and one can easily imagine how short-lived a bridge or a steel rail would be if it were strained

*Because of the difficulty of locating the point *p* accurately.

beyond the elastic limit by every passing train. The total elongation of a sample of steel at the breaking point b, Fig. 95, is an important indication of the toughness of the steel. The following table gives the important properties of several grades of steel.

TABLE.

Physical properties of steel.

Carbon content in per cent.	Elastic limit pounds per square inch.	Tensile strength pounds per square inch.	Elongation in 4 inches in per cent.	Stretch modulus in pounds per square inch.
0.17	51,000	68,000	33.5	29,800,000
0.55	57,000	106,100	16.2	"
0.82	63,000	142,250	8.5	"

93. Resilience. The work done per unit volume in straining a substance to its elastic limit is called the *resilience* of the substance. This work per unit volume is equal to one half the product of the limiting stress and the limiting strain, according to equation (49), and it is represented by the area under the straight portion of the curve in Fig. 95. For example, the resilience of 0.82-per-cent. carbon steel is

$$\tfrac{1}{2} \times 63{,}000 \frac{\text{pounds}}{\text{inch}^2} \times 0.0021 = 66.1 \frac{\text{inch-pounds}}{\text{inch}^3} = 5.5 \frac{\text{foot-pounds}}{\text{inch}^3}$$

that is, 5.5 foot-pounds per cubic inch. Thus it would take 100 cubic inches of this steel (about 25 pounds) made into a spring to store sufficient energy to supply one horse-power for one second, provided the spring could be so designed as to be strained in every part to its elastic limit when wound up. The resilience of spring steel may be as high as 10 or 12 foot-pounds per cubic inch, the resilience of good cast iron is about 0.5 foot-pound per cubic inch.

The resilience of a substance is a measure of its strength to withstand a sudden shock, inasmuch as a blow of a hammer, for example, bends a bar until the kinetic energy of the hammer is all used in bending the bar. A structure subject to shocks should be made of highly resilient material.

94. Elastic hysteresis. In nearly all substances there is more or less of a tendency for strain to persist after the stress has ceased. This is of course very markedly the case where a substance is strained beyond the elastic limit, but in many substances the elastic limit is by no means sharply defined, and very slight strains do not entirely disappear when the stress ceases. When a substance is subjected to a stress which increases and decreases periodically between two limiting values S_1 and S_2, the relation between stress and strain is somewhat as indicated in Fig. 96, where ordinates represent stress and abscissas represent strain. The branch *a* of the curve represents the relation between stress and strain while stress is increasing, and the branch *b* represents the relation between stress and strain while stress is decreasing. This divergence of the curve of stress and strain for increasing and decreasing stress is called *elastic hysteresis*. The increasing and decreasing stress is here supposed to increase and decrease very slowly. If the increase and decrease is rapid the divergence of the two curves *a* and *b*, Fig. 96, is due to hysteresis and also to elastic lag.

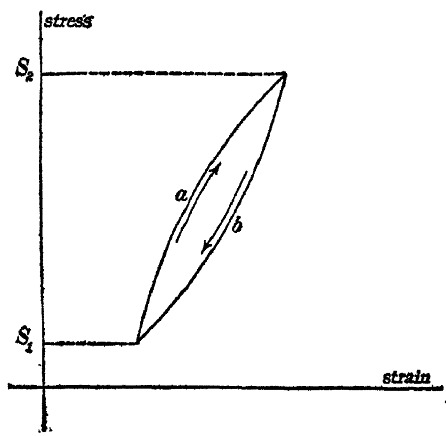


Fig. 96.

95. Elastic lag; viscosity. Many substances, glass for example, when subjected to stress, take on a certain amount of strain quickly, after which the strain increases slowly for a time; and when the stress is relieved, a remnant of the strain persists for a time. This phenomenon is called *elastic lag*.

The strain of some substances, such as pitch, continues to increase indefinitely, although it may be very slowly, when they

are under stress. Such substances are said to be *viscous*. Nearly all metals are viscous when subjected to great stress.

Elastic hysteresis, elastic lag, and viscosity cause energy to be dissipated in a substance when it is strained. Thus the vibrations of a steel spring die away rapidly even in a vacuum, on account of the conversion of energy into heat as the spring is repeatedly distorted.

96. Elastic fatigue. The repeated application of a stress weakens a metal so that it will break under less than its normal breaking stress, or less even than the stress corresponding to its

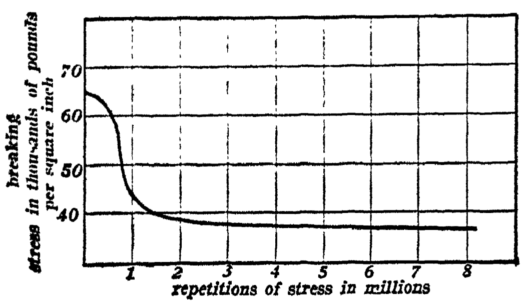


Fig. 97.

elastic limit. Fig. 97 shows the decrease in tensile strength of a sample of mild steel with repetitions of the stress.

Continued repetition of stress causes an increase in the amount of energy dissipated by elastic lag and viscosity. Thus the vibrations of a torsion pendulum die away faster after it has been kept vibrating for several days, than at first.

HYDROSTATIC PRESSURE AND ISOTROPIC STRAIN.

97. Hydrostatic pressure.* A fluid at rest not only pushes normally against a surface which is exposed to its action, but two contiguous portions of a fluid at rest always push on each other at right angles to a small plane q which may be imagined to separate them as indicated in Fig. 98. Whatever the *direction* of the small plane q may be, the *force action per unit area* across

*Hydrostatic pressure is discussed also in the chapter on hydrostatics.

it is the same. This fact was first pointed out by Pascal (1623–1662) and it is sometimes called Pascal's principle.* The force action per unit area at a point in a fluid is generally represented by the letter p and it is called the *hydrostatic pressure* at the point.

98. Isotropic strain. When a substance like glass or cast metal is subjected to an increase of hydrostatic pressure the substance is reduced in size without being changed in shape; such a strain is called an *isotropic strain.* Let V be the original volume of the substance, and let v be the diminution of volume due to the increase of pressure. It is convenient to express v as a fraction of V, and this fraction v/V is used as a measure of the isotropic strain.

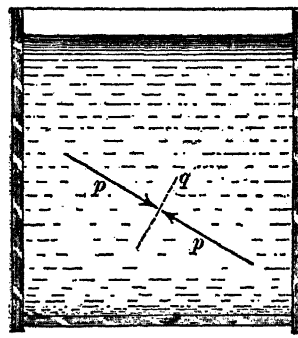


Fig. 98.

99. Bulk modulus of a substance. The diminution of volume of a substance per unit of original volume (v/V) is proportional to the increase of hydrostatic pressure, except, of course, when the increase of pressure is very great and the decrease of volume considerable. Therefore, for small changes of volume, the ratio of increase of hydrostatic pressure to decrease of volume per unit volume is a constant for a given substance (at a given temperature). That is

$$B = \frac{p}{\frac{v}{V}} = \frac{pV}{v} \tag{51a}$$

in which V is the volume of a substance at a given pressure, v is the decrease of volume due to an increase of pressure p, and B is a constant for the given substance. This quantity B is called the *bulk modulus* of the substance. The reciprocal of B is called

*This principle may be established by considering the equilibrium of a small portion or element of a fluid bounded by the three reference planes XY, YZ and ZX and by a diagonal plane.

the *coefficient of compressibility* of the substance. Therefore, writing C for $1/B$ equation (51)a becomes

$$C = \frac{v}{pV} \qquad (51)b$$

or

$$v = pVC \qquad (51)c$$

The coefficient of compressibility of a substance is the change of volume per unit original volume per unit increase of pressure, and, according to equation (51)c, the decrease of volume of a substance due to a given increase of pressure is equal to the product of the increase of pressure, the original volume, and the coefficient of compressibility.

TABLE.*

Coefficients of compressibility at 20° C. for moderate increase of pressure.

(Decrease of volume per unit volume per atmosphere increase of pressure.)

SUBSTANCE.	$C \times 10^6$.
Ether	170.0
Alcohol	101.0
Water	46.0
Glass	2.2
Steel	0.68

100. Potential energy of isotropic strain. The potential energy per unit volume of an elastic substance under increased pressure is equal to one half the product of the increase of hydrostatic pressure p and the strain v/V, or it is equal to one half the product of the bulk modulus B and the square of the strain (v^2/V^2). This relation may be derived in a manner very similar to the proof of equations (49) and (50).

101. Compressibility of gases. Boyle's law. Solids and liquids generally decrease but slightly in volume when subjected to increase of pressure. Thus the volume of water decreases about one ten-thousandth part when subjected to an increase of

*See *Physikalish-Chemische Tabellen* by Landolt and Bornstein, Berlin, 1895 for a very complete collection of data of all kinds.

pressure of 30 pounds per square inch, and the volume of steel decreases about one ten-thousandth part when subjected to an increase of pressure of 2,000 pounds per square inch.

Gases, on the other hand, decrease greatly in volume when subjected to increase of pressure. The remarkable contrast between water and air in regard to compressibility may be shown by filling a bicycle pump with air and then with water, and striking the piston rod in each case with a hammer. The air will be found to act as a cushion, and the water will appear to be as solid as if the whole pump barrel and piston were one piece of steel. When a steam engine is started, the water which usually collects in the steam pipes may enter the cylinder in sufficient quantity to cause the moving piston to burst the cylinder head.

When the temperature of a gas is kept at a constant value, the volume of the gas is inversely proportional to the pressure to which the gas is subjected. That is

$$v = \frac{k}{p}$$

or

$$pv = k \tag{52}$$

in which v is the volume of a given amount of gas, p is the pressure of the gas, and k is a constant. This relation, which is known as Boyle's law, was discovered by Robert Boyle* in 1660, and more completely established by Mariotte, who discovered it independently in 1676. It is very accurately true of such gases as hydrogen, nitrogen and oxygen at ordinary temperatures and pressures, but all gases deviate from it appreciably, especially at low temperatures and under great pressures. See the discussion of the properties of gases in the chapters on heat.

SHEARING STRESS AND SHEARING STRAIN.

102. Shearing stress. The type of stress and strain in a twisted rod is called shearing stress and shearing strain, and the discussion of this type of stress and strain is somewhat obscured by the fact that there is no familiar example in which homogeneous shearing stress and shearing strain occur; the stress and strain in a twisted

***New Experiments touching the Spring of Air*, Oxford, 1660.

rod are non-homogeneous. Any intelligible discussion of shearing stress and shearing strain must, however, be based on a case in which the stress and strain are homogeneous. Consider, therefore, a cubical portion of a substance $ABCD$, Fig. 99, and suppose that outward force (S units of force per unit of area) act upon the faces AB and CD, that inward force (S units of force per unit of area) act upon the faces AC and BD, and that no force at all acts on the two faces of the cube which are parallel to the plane of the paper. Then the material of the cube will be subject to what is called a *shearing stress*, and the stress will be homogeneous. The character of the force action between contiguous parts of the material of the cube is as

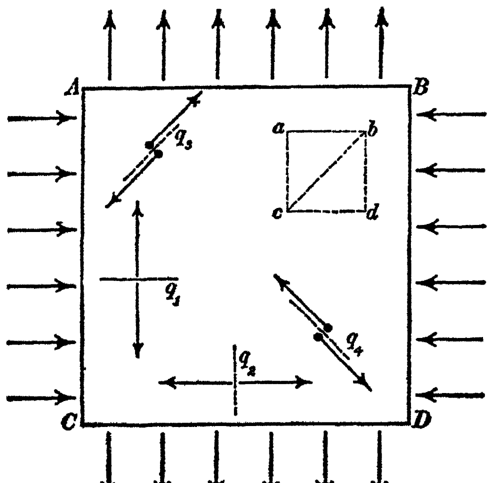


Fig. 99.

follows: A *pull* of S units of force per unit area acts across any plane q_1 which is parallel to the faces AB and CD of the cube; a *push* of S units of force per unit area acts across any plane q_2 which is parallel to the faces AC and BD of the cube; a sliding force, or tangential force, as it is called, of S units of force per unit of area acts across any plane q_3 or q_4 which is parallel to the diagonal plane AD or BC; and no force at all acts across a plane which is parallel to the plane of the paper.

To show that the force action across the diagonal planes q_3 and q_4 is a tangential force action and that the tangential or sliding force is S units of force per unit area, consider any unit cube $abcd$ of the material. The area of each face of this cube is unity, and the area of the diagonal plane bc is $\sqrt{2}$ units. The total force acting on the face bd is a push of S units, the total force acting on the face cd is a pull of S units, and the resultant of these two forces is a force parallel to bc and equal to $\sqrt{2}\cdot S$ as shown in Fig. 100. Similarly, the resultant of the forces acting on the faces ab and ac is a force parallel to cb and equal to $\sqrt{2}\cdot S$. Therefore, the force action across the diagonal plane bc is a tangential force action equal to $\sqrt{2}\cdot S$, which, divided by the area of the diagonal plane bc, gives a tangential force action of S units of force per unit of area.

It can be shown in the same way that the force action across q_4, Fig. 99, is a tangential force of S units per unit of area. It is on account of the purely tangential forces across q_3 and q_4 that this type of stress is called a shearing stress.

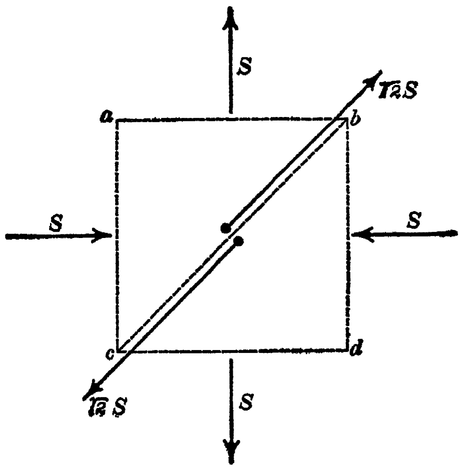


Fig. 100.

103. Shearing strain. The effect of the prescribed stress in Fig. 99 is to shorten the cube in the direction of the push and to lengthen the cube by an equal amount

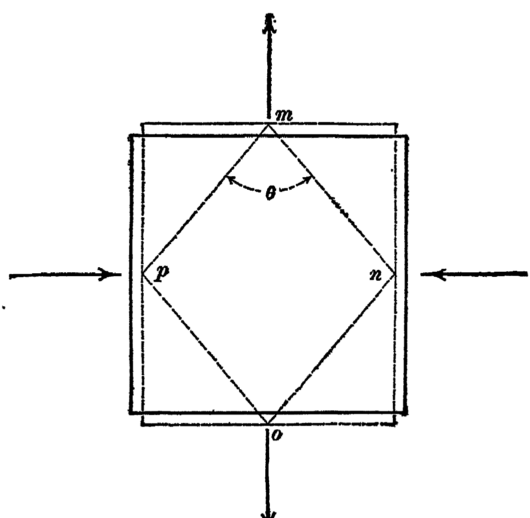


Fig. 101.

in the direction of the pull, without changing the dimensions of the cube in a direction at right angles to the plane of the paper in Fig. 99. This distortion is represented by the dotted rectangle in Fig. 101, and the dotted rhombus *mnop* in Fig. 101 is a figure which was square in the unstrained material. The angle θ is less than $90°$ and the value of the angle $90°-\theta$ (expressed as a fraction of a radian) is called the *angle of the shearing strain*, or simply the *angle of shear*. It is usually represented by the letter ϕ, and it is used as a measure of the shearing strain.

Let L be the original length of each edge of the cube in Fig. 99, and let l be the increase of length in the direction of the pull and the decrease of length in the direction of the push. It is convenient to express l as a fraction of L, and we will represent this fraction by the letter α $(=l/L)$. It is important to know that the angle of shear ϕ as above defined is equal to 2α. That is

$$\phi=2\alpha \tag{53}$$

The full-line square in Fig. 102 is the figure which when distorted becomes the rhombus in Fig. 101, and the small triangle in Fig. 102 is enlarged in Fig. 103. The

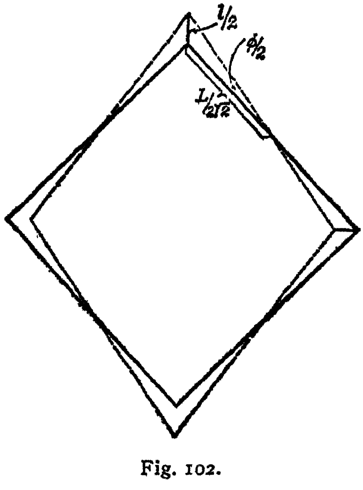


Fig. 102.

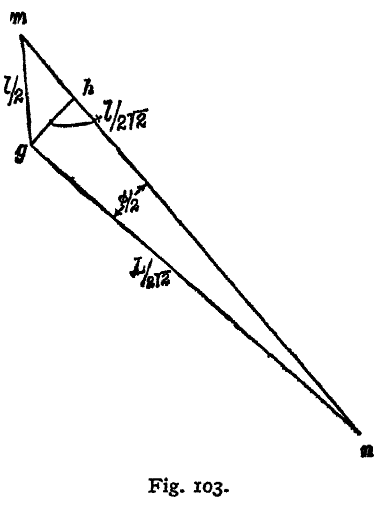


Fig. 103.

angle $\phi/2$ is very small and it is therefore sensibly equal to *gh* divided by *gn*. That is, $\phi/2$ is equal to l/L $(=\alpha)$ so that $\phi=2\alpha$.

104. Slide modulus of a substance. The angle of shear ϕ produced in Fig. 99 by the shearing stress S is proportional to S, according to Hooke's law, so that the ratio S/ϕ is a constant for a given substance, within the limits of elasticity. This constant is called the *slide modulus* of the substance and it is represented by the letter n. Therefore we have

$$n=\frac{S}{\phi} \tag{54}$$

The slide modulus of a substance is sometimes called the *shearing modulus* of the

substance. It is approximately equal to $\frac{3}{8}$ of the stretch modulus (Young's modulus) for metals.

105. Discussion of a twisted rod. Consider a cylindrical rod of radius R and length L, and suppose that one end of the rod is fixed while the other is turned through the angle θ so as to twist the rod. Consider a cylindrical shell of the material of the rod of which the radius is r, and imagine this cylindrical shell to be cut along one side and laid out flat so that it may be pictured on a flat surface.* Figures

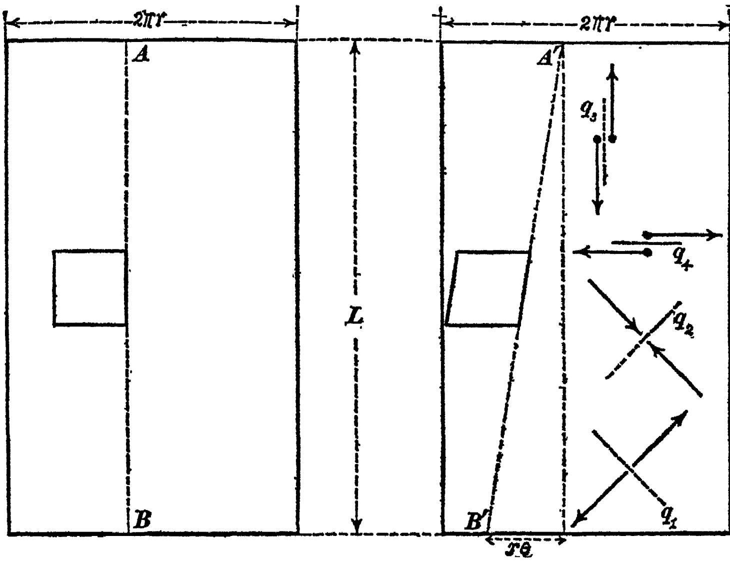


Fig. 104. Fig. 105.

104 and 105 represent the cylindrical shell laid out flat, or developed, in this way; Fig. 104 before twisting, and Fig. 105 after twisting. The line AB which is parallel to the axis of the rod in the untwisted rod takes the position $A'B'$ after the rod is twisted, and the small square becomes a rhombus. The portions of the rod in the cylindrical shell under consideration are subjected to a shearing strain of which the angle of shear is

$$\phi = \frac{r\theta}{L} \tag{55}$$

*This discussion of the strain in a twisted rod may be made more easily intelligible by means of a model as follows: A tin cylinder about 20 cm. in diameter and 35 cm. long, has wooden disks fixed in each end. On one end is an additional wooden disk which turns on a nail which is the axis of the cylinder. Loosely woven muslin is tacked at one end to the movable disk and at the other end to the fixed disk. This muslin fits the tin cylinder closely, and the seam at one side is sewed. On this muslin a small square may be drawn like Fig. 104, and also small circles, and when the movable disk is turned through a considerable angle, the distortion of the square and of the circles will give a clear idea of the character of the strain in a cylindrical shell of a twisted rod.

as is evident from Fig. 105. The character of the force action between contiguous portions of the material of the rod may be understood by comparing Fig. 105 with Fig. 99. There is a tangential force action across every vertical plane q_3 and across every horizontal plane q_4;* there is a normal pull across every plane like q_1, and a normal push across every plane like q_2; and the force per unit area, S, in each case is equal to $n\phi$, according to equation (54). Therefore, substituting for ϕ its value from equation (55), we have

$$S = \frac{nr\phi}{L} \tag{56}$$

The tangential stress across vertical planes like q_3, Fig. 105, is concentrated at the bottom of a sharp groove cut in the rod parallel to its axis, like a key seat in a shaft, and such a groove therefore weakens the rod very much indeed.

Constant of torsion of a rod or wire. The total force action across a complete section of a twisted rod is a torque T about the axis of the rod and the value of the torque is

$$T = -\frac{\pi n R^4 \theta}{2L} \tag{57}$$

in which n is the slide modulus of the material of the rod or wire, R is the radius of the rod or wire, L is the length of the rod or wire, and θ is the angle through which one end of the rod or wire is twisted. The negative sign is written for the reason that the torque tends to reduce θ,† that is, T and θ are opposite in sign. This equation (57) shows that T is proportional to θ, and the proportionality factor $\pi n R^4/2L$ is called the *constant of torsion* of the wire or rod. If the constant of torsion of a rod or wire is determined by observing the angle of twist θ produced by a known torque, the slide modulus of the material may be calculated from equation (57).

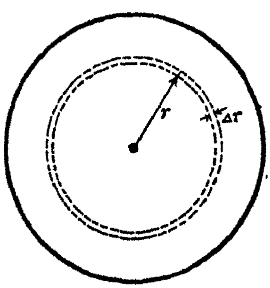


Fig. 106.

Proof of equation (57). Let Fig. 106 represent a sectional view of the rod. Consider the narrow annulus of width Δr and radius r, as shown by the dotted lines. The force action per unit area across this annulus is $rn\theta/L$ according to equation (56), and this force action is at right angles to r at each point of

*The force actions between contiguous portions of a twisted rod are inferred from the character of the distortion at each point as represented in Fig. 105. These force actions may be made clearly evident by the use of two models, as follows: (*a*) A bundle of smooth square pine sticks bound together, and, if desired, turned to a cylindrical shape, shows sliding along vertical planes like q_3, Fig. 105, when the bundle of sticks is twisted. (*b*) A brass tube with a slit along a helical line which is at each point inclined at an angle of 45° to the axis of the tube, is as strong as an uncut tube to withstand a twist in one direction (the faces of the slit push against each other normally), but the slit opens when the tube is twisted in the other direction.

†The reacting torque of the twisted rod is here referred to.

the annulus. Therefore the torque action, about the axis of the rod, of the force which acts across unit area of the annulus is $r \times rn\theta / L$, which, multiplied by the area $2\pi r \cdot \Delta r$ of the annulus, gives the total torque action ΔT across the annulus. That is

$$\Delta T = \frac{2\pi r^3 n\theta}{L} \cdot \Delta r$$

or

$$T = \frac{2\pi n\theta}{L} \int_0^R r^3 dr = \frac{\pi n R^4 \theta}{2L}$$

GENERAL EQUATIONS OF STRESS AND STRAIN.

106. Principal stretches of a strain. A small spherical portion of a body always becomes an ellipsoid when the body is distorted.* A distortion which changes a sphere into an ellipsoid consists always of simple increase or decrease of linear dimensions in three mutually perpendicular directions. These mutually perpendicular directions are called the *axes of the strain*, and the increase of length per unit length (l/L) in the directions of the respective axes are called the *principal stretches* of the strain. The principal stretches of a strain are represented by the letters f, g, and h.

107. Principal pulls of a stress. Imagine a small plane area q, called a *section*, in the interior of a body under stress. The portions of the body on the two sides of this section exert on each other a definite force in a definite direction, and the force per unit area of the section is called the *stress on the section*. When the force is normal to the section, the stress on the section is called a *pull*, positive or negative as the case may be. When the force is parallel to the section, the stress on the section is called a *tangential stress*.

The conditions of equilibrium of a small portion of a body under stress require* that at a point in the body there be three mutually perpendicular sections across which the force action is normal, or on which the stress is a pull. These three sections are called the *principal sections of the stress*, the three lines perpendicular to them are called the *axes of the stress*, and the pulls on the three sections are called the *principal pulls of the stress*. These three pulls constitute the stress at the point and if these pulls are given the stress is completely specified.

108. General equations of stress and strain. Let F be a longitudinal stress, that is, a simple pull, in the direction of the x-axis of reference. Such a stress causes a stretch aF in the direction of the x-axis, and a negative stretch $-bF$ in the directions of the y and z-axes. Therefore, writing f', g' and h' for the three stretches due to the simple pull F, we have

$$\begin{aligned} f' &= aF \\ g' &= -bF \\ h' &= -bF \end{aligned} \tag{i}$$

Similarly, let G be a longitudinal stress in the direction of the y-axis of reference, and let f'', g'', and h'' be the three stretches, parallel to x, y, and z axes respectively, produced by G. Then we have:

*See *Elasticity, theory of*, in Encyclopedia Britannica, 9th Ed.

$$\begin{aligned} f'' &= -bG \\ g'' &= aG \\ h'' &= -bG \end{aligned} \tag{ii}$$

Similarly, let H be a longitudinal stress parallel to the z-axis of reference, and let f''', g''', and h''' be the three stretches, parallel to the x, y, and z axes, respectively, produced by H. Then we have:

$$\begin{aligned} f''' &= -bH \\ g''' &= -bH \\ h''' &= aH \end{aligned} \tag{iii}$$

Experiment shows* that the stretch produced in any direction by a number of pulls acting together is equal to the sum of the stretches in that direction produced by the respective pulls acting separately. Therefore:

$$\begin{aligned} f &= f'+f''+f''' \\ g &= g'+g''+g''' \\ h &= h'+h''+h''' \end{aligned} \tag{iv}$$

in which f, g, and h are the stretches, parallel to the x, y, and z axes, respectively, produced by the three pulls F, G, and H acting together. Therefore substituting the values of f', f'', f''', g', g'', g''', h', h'', and h''' from equations (i), (ii), and (iii) in equation (iv), and we have:

$$\begin{aligned} f &= aF-bG-bH \\ g &= -bF+aG-bH \\ h &= -bF-bG+aH \end{aligned} \tag{58}$$

These equations give the strain (f, g, and h) which is produced in an isotropic elastic solid by any stress (F, G, and H), and it shows that an isotropic elastic solid has but two constants of elasticity a and b. In fact the quantity a is equal to $1/E$ and the quantity b is equal to σa, where σ is the value, β'/β, of Poisson's ratio. See Art. 87. Starting with these relations, it is very easy to derive expressions for the bulk modulus B and for the slide modulus n of a substance in terms of E and σ by using equations (58); considering that $F=G=H=p$ in the case of hydrostatic pressure, and that $F=+S$, $G=-S$, and $H=0$ in the case of a shearing stress.

PROBLEMS.

118. A helical spring is elongated by an amount of 1.2 inches when a 4-pound weight is hung upon it. How much additional elongation is produced by 1 pound additional weight? By two

*In general, any effect which is *proportional* to a cause, may be resolved into parts which correspond to the parts of the cause. Thus a spring stretches in proportion to the stretching force. One kilogram produces, say, one centimeter elongation; two kilograms produce two centimeters elongation, which is one centimeter for each kilogram. See Art. 32.

pounds additional weight? By three pounds additional weight? By four pounds additional weight? Ans. 0.3 inch; 0.6 inch; 0.9 inch; 1.2 inch.

Note. Assume in this problem, and in those that follow, that the elastic limit is not exceeded.

119. The middle of a long beam is depressed 2 inches by a load of 5,000 pounds. How much will it be depressed by a load of 15,000 pounds? Ans. 6 inches.

120. A long rod is fixed at one end, and a twisting force, or torque, of 100 pound-inches applied at the free end causes the free end to turn through an angle of 10°. What torque would be required to turn the free end of the rod through 26°? Ans. 260 pound-inches.

121. A rod 2 inches in diameter and 20 feet long is stretched to a length of 20 feet and ¼ inch by a force of 10,000 pounds-weight. What is the value of the longitudinal stress, and what is the value of the longitudinal strain? Ans. Stress 3,183 pounds per square inch; strain 0.001042.

122. A rod 20 feet long and 1 inch in diameter is subjected to a pull of 20,000 pounds per unit of sectional area causing it to be lengthened to 20.02 feet, that is one part in a thousand, and causing it to contract to a diameter of 0.99975 inch, that is, 25 parts in one hundred thousand. What is the length and what is the diameter of the rod when it is subjected to a pull of 40,000 pounds per unit sectional area? Ans. Length 20.04 feet, diameter 0.9995 inch.

123. A wire 200 inches long and 0.1 inch in diameter is pulled with a force of 150 pounds. The elongation produced is ½ inch. What is the value of the stretch modulus of the material? Ans. 7,640,000 pounds per square inch.

124. A wire five feet long and 0.06 square inch sectional area is subjected to a stretching force of 300 pounds. The stretch modulus of the material is 28,000,000 pounds per square inch. What elongation is produced? Ans. 0.0107 inch.

125. A steel beam is bent so that its middle line forms the arc

of a circle 600 inches in radius. What is the elongation per unit length of a filament 2 inches from the middle line? Ans. 0.00333.

126. The stretch modulus of the steel of which the beam of the previous problem is made is 30,000,000 pounds per square inch. What is the pull (force per unit area of course) of a filament of the beam 2 inches from the middle line of the beam? Ans. 100,000 pounds per square inch.

127. What is the resilience of spring steel of which the elastic limit is 70,000 pounds per square inch and of which the stretch modulus is 30,000,000 pounds per square inch? Ans. 6.8 foot-pounds per cubic inch.

128. A cork ½ inch in diameter is pushed with a force of 20 pounds-weight into a bottle which is completely filled with water. What hydrostatic pressure is produced in the bottle? Neglect the friction of the cork against the glass neck of the bottle. Ans. 101.8 pounds per square inch.

129. A body subjected to hydrostatic pressure is decreased in length, in breadth, and in thickness by 5 parts in a thousand (initial). By how many parts per thousand (initial) is the volume reduced? Ans. 15 parts per thousand.

130. What is the value of the bulk modulus of water when 2,000 cubic inches of water are reduced to 1,880 cubic inches by an hydrostatic pressure of 3,000 lbs. per square inch? Ans. 50,000 pounds per square inch.

131. Calculate the compressibility of water from the answer to the previous problem and explain its meaning. Ans. 0.00002 square inch per pound.

132. A bicycle pump is full of air at 15 pounds per square inch, length of stroke 12 inches; at what part of the stroke does air begin to enter the tire at 40 pounds per square inch above atmospheric pressure? Assume the compression to take place without rise of temperature. Ans. When piston is 3.27 inches from the end of the cylinder.

133. The clearance space behind the piston of an air compressor when the piston is at the end of its stroke is $\frac{1}{50}$ of the *volume*

swept by piston during the stroke. What is the greatest pressure that can be produced in a compressed air reservoir by this compressor, the compression of the air in the cylinder being assumed to be without change of temperature? Ans. Fifty times the initial pressure.

Note. As a matter of fact the air in an air compressor is heated very consider ably by the compression.

134. The piston of an air pump is 0.01 inch from the bottom of the cylinder when it is at the end of its stroke, and the pressure of the air in the clearance space is then at atmospheric pressure. The length of stroke is 6 inches. What is the highest vacuum which can be produced by the pump? Ans. 1/600 of an atmosphere.

135. A cubical piece of steel is shortened two parts in a thousand in one direction, lengthened two parts in a thousand in a direction at right angles to the first, and unchanged in dimension in the third direction, as represented in Fig. 101. What is the value of the angle of shear in degrees? Ans. 0.229 degree.

136. A steel rod 120 inches long is fixed at one end and the other end is turned through 5 degrees of angle. Consider a small portion p of the rod at a distance of 1 inch from the axis of the rod. Find the angle of shear ϕ of this small portion p of the metal. Ans. 0.0417 degree.

137. The slide modulus of the steel used in the rod of problem 136 is 12 million pounds per square inch. Find the shearing stress in the small portion p of the rod. Ans. 8,735 pounds per square inch.

138. A steel shaft 500 inches long and 3 inches in diameter transmitting 100 horse-power is subject to a torque of 23,100 pound-inches of torque. The slide modulus of the material of the shaft is 12,000,000 pounds per square inch. Calculate the angle through which one end of the shaft is twisted relative to the other end. Ans. 0.121 radians or 6.94 degrees.

139. The three stretches of a strain are $+0.015$, $+0.025$ and -0.025. What are the semi-axes of the ellipsoid into which a sphere 10 inches in radius is distorted by this strain? Strain supposed to be homogeneous. Ans. 10.15 inches, 10.25 inches, 9.75 inches.

140. A force of 250 pounds acts across a section of which the area is $\frac{1}{4}$ square inch. What is the value of the stress on the section? Ans. 1,000 pounds per square inch.

141. A square rod $2 \times 1\frac{1}{2}$ inches is subjected to a tension of 75,000 pounds. What kind of stress acts across a section of the rod and what is its value? Ans. A normal stress of 25,000 pounds per square inch.

142. Two long strips of metal are lapped and fastened by a single rivet of which the sectional area is two square inches. The two strips are subjected to a tension of 10,000 pounds. What kind of stress acts across the middle section of the rivet and what is the value of the stress? Ans. A tangential stress of 5,000 pounds per square inch.

143. Derive the equation expressing the bulk modulus of a substance in terms of its stretch modulus and Poisson's ratio. The stretch modulus of steel is 30 million pounds per square inch, and Poisson's ratio for steel is 0.28. Find the value of the coefficient of compressibility and compare it with the value given in the table in Art. 99. One atmosphere is equal to 14.7 pounds per square inch. Ans. $B=E/[3(1-2\sigma)]$; $C\times 10^6=0.65$; from table $C\times 10^6=0.68$.

144. Derive the equation expressing the slide modulus of a substance in terms of its stretch modulus and Poisson's ratio, and calculate the slide modulus of steel using the data given in problem 143. Ans. $n=E/[2(1+\sigma)]$; $n=11.72\times 10^6$ pounds per square inch.

CHAPTER VIII.

HYDROSTATICS.

109. Pressure at a point in a fluid.* The force with which a fluid at rest pushes against an element of an exposed surface is at right angles to the element and proportional to the area of the element. The *force per unit area* is called the *hydrostatic pressure* or simply the *pressure* of the fluid at the place where the element of area is located and it is usually represented by the letter p. When the pressure has the same value throughout a fluid the pressure is said to be uniform, when the pressure varies from point to point in a fluid the pressure is said to be non-uniform. When the pressure in a fluid is uniform the total force F acting on an exposed plane surface is

$$F = pa \tag{59}$$

where a is the area of the surface.†

Examples. (*a*) *Steam pressure.* The piston of a steam engine is pushed by a force equal to pa, where a is the area of the piston, and p is the pressure of the steam in the cylinder. Every part of the inside surface of a steam boiler is pushed outwards by the steam.

(*b*) *Atmospheric pressure.* The force with which the air pushes on the surfaces of bodies does not ordinarily appeal to our senses. It is shown however by the collapse of a thin-walled vessel when the inside pressure is reduced by pumping out the air. Atmospheric pressure is also strikingly shown by means of the apparatus known as the *Magdeburg Hemispheres.* This consists of two metal cups which fit together air tight and form a hollow vessel from which the air may be removed by pumping.

*The term *fluid* includes liquids and gases, as explained in Art. 83.

†See Art. 97.

The pressure of the outside air then holds the cups together and a considerable effort is required to separate them. This celebrated experiment was devised by Otto von Guerike, the inventor of the air pump, and it was performed publicly in Magdeburg in 1654.

(*c*) *The hydrostatic press* consists essentially of a strong cylinder with a large plunger or piston, and a pump with a small piston or plunger for forcing water into the large cylinder under high pressure. The great forging press at the Bethlehem Steel Works has two plungers each fifty inches in diameter, thus exposing a total of about 3,600 square inches of piston area to the water, which is forced into the cylinders of the press at a pressure of 8,000 pounds per square inch. This gives a total force of about 14,000 tons upon the two plungers.

Pascal's principle. The force per unit area which is exerted by a fluid on an exposed surface at a given place in the fluid is independent of the direction of the surface. Thus the air pushes against an exposed surface with about 15 pounds of force per square inch whether the surface be horizontal or vertical or inclined at any angle.

110. The circumferential tension in the walls of a cylindrical pipe. The pressure of a fluid in a cylindrical pipe produces a

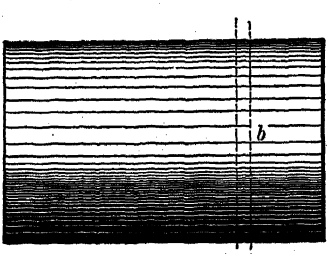


Fig. 107.

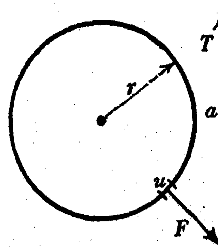


Fig. 108.

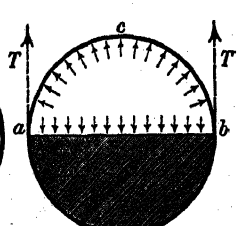


Fig. 109.

tension in the material of the pipe. Consider a narrow band b, Fig. 107, of the material of a pipe, the width of the band being one inch. An end view of this band is shown in Fig. 108, and, since the band is one inch wide, each inch of its circumference is

pushed outwards by a force equal to p pounds, where p is the steam or water pressure in pounds per square inch. Therefore, according to Art. 60, the circumferential tension in the band b is equal to rp pounds per inch of width, where r is the radius of the pipe in inches.

It is instructive to establish this result from another point of view as follows: Imagine the cylindrical pipe to be half solid, as shown by the shaded area in Fig. 109, then, considering one inch of length of the pipe as before, the area of the flat surface ab is $2r$, the force acting on this flat face is $2rp$, and this force is balanced by the two forces TT, so that the value of each force T is equal to rp.

Since the circumferential tension in a cylindrical pipe is equal to rp, it is evident that a small pipe can withstand a much greater pressure than a large pipe, the thickness of the walls of the pipes being the same.

Longitudinal tension in a boiler shell. Consider a cylindrical boiler of radius r. The area of each end of the boiler is πr^2, the outward force exerted on each end of the boiler is $\pi r^2 p$, and this force produces lengthwise (longitudinal) tension in the boiler shell. Inasmuch as the width of the boiler shell which withstands these endwise forces is equal to the circumference of the boiler $(2\pi r)$, it is evident that the endwise tension is $\pi r^2 p/(2\pi r)$ or $\frac{1}{2}pr$ units of force per unit width.

111. Pressure in a liquid due to gravity. The pressure in a fluid under the action of gravity increases with the depth. If the density of the fluid is the same throughout, and this is approximately the case in any liquid, then the pressure at a point distant x beneath the surface of the liquid *exceeds the pressure at the surface* by the amount

$$p = xdg \tag{60a}$$

in which p is expressed in dynes per square centimeter when the density d of the liquid is expressed in grams per cubic centimeter, the distance x in centimeters and the acceleration of

gravity g in centimeters per second per second; or p is expressed in poundals per square foot if the density of the liquid d is expressed in pounds per cubic foot, the distance x in feet and the acceleration of gravity g in feet per second per second.

The most useful form of the above equation is that which gives the pressure in pounds-weight per square foot, namely

$$p = xd \tag{60b}$$

in which p is expressed in pounds per square foot, x is expressed in feet and d is the density of the liquid in pounds per cubic foot.

Discussion of equation (60). The force with which a liquid pushes on an element of an exposed surface is independent of the direction of the surface element according to Pascal's principle. Therefore we may derive equation (60) by considering a horizontal surface a square feet in area exposed to the action of the liquid

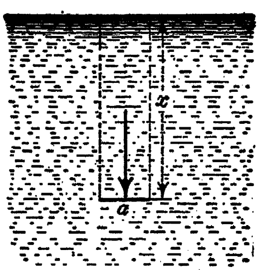


Fig. 110.

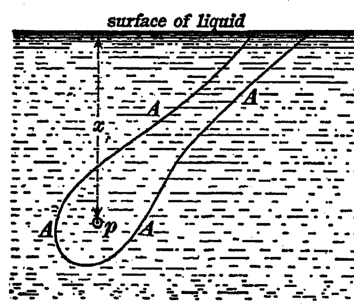


Fig. 111.

as shown in Fig. 110. The volume of the liquid directly above a is ax cubic feet, the mass of this portion of liquid is axd pounds where d is the density of the liquid in pounds per cubic foot, the force in poundals with which gravity pulls on this portion of the liquid is $axdg$, and therefore the total force with which this portion of liquid pushes down on the element a is equal to $axdg$ poundals, so that the force per unit area is $axdg$ divided by a, or xdg poundals per square foot.

Equation (60) involves no consideration of the shape of the vessel which contains the liquid. As a matter of fact, the pres-

sure at a point in a liquid exceeds the pressure at the surface of the liquid by the amount xdg whatever the shape and size of the containing vessel may be. This may be made almost self-evident as follows: Given a point p, Fig. 111, at a distance x beneath the surface of a large body of liquid. *Imagine a portion of the liquid AAAA, of any shape whatever, extending from p to the surface.* The liquid surrounding the portion $AAAA$ acts on $AAAA$ exactly as a containing vessel of the same shape would act, and therefore the pressure of p is exactly what it would be if the portion $AAAA$ were contained in such a vessel.

112. The total force acting on a water gate and its point of application. When a plane surface of area a is exposed to the action of fluid under uniform pressure, the total force acting on the surface is pa and the point of application of this force is the center of figure of the exposed plane surface. When, however, a plane surface

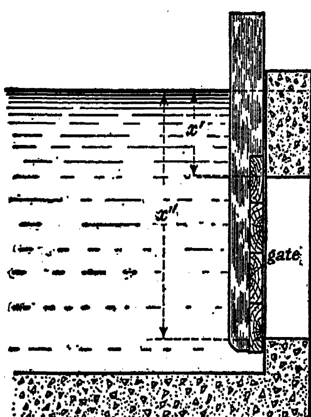


Fig. 112*a*.

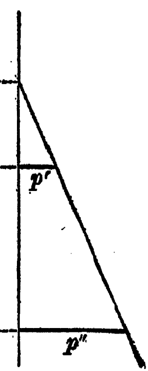


Fig. 112*b*.

is exposed to the action of the fluid in which the pressure is not uniform, the total force is, of course, *not* equal to pa, for p has different values at different parts of the surface, and the point of application of the total force is *not* at the center of figure of the exposed surface. The simplest case is that in which the water in a tank pushes against the rectangular side of the tank, or the case in which water pushes against a rectangular gate as shown in Fig. 112*a*. The pressure at the top of the gate is $p'=0.434x'$ pounds per square inch, the pressure at the bottom of the gate is $p''=0.434x''$, the average pressure over the whole gate is $(p'+p'')/2$ or $0.434(x'+x'')/2$ pounds per square inch, and the total force F acting on the gate is equal to the product of this average pressure and the area of the gate in square inches.

The point of application of the total force with which the water pushes on the gate is the point at which a single force F', Fig. 113, could be applied to balance the push of the water. This point is evidently below the center of the gate; in fact the distance X, Fig. 113, is

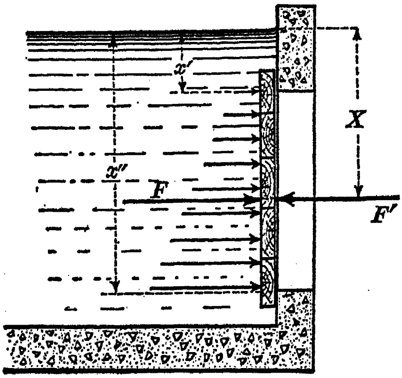


Fig. 113.

$$X = \frac{2}{3}\left(\frac{x''^3 - x'^3}{x''^2 - x'^2}\right) \qquad (61)$$

In the case of the side of a rectangular tank, or in case of a dam (where x' equals zero) the distance from the surface of the water to the point of application of the total force which pushes on the side of the tank or against the dam is two thirds of the depth of the water.

Proof of equation (61). The total force F, Fig. 113, is equal to the area of the gate $w(x''-x')$ multiplied by the average pressure $0.434(x''+x')/2$, w being the horizontal width of the gate; and the torque action of F about any conveniently chosen point is equal to the sum of the torque actions, about the same point, of the forces acting on the various elements of the surface of the gate. Consider a horizontal strip of the gate distant x beneath the surface of the water and of which the vertical breadth is dx. The force acting on this strip is $0.434x \times w dx$, and the torque action of this force about a point at the surface of the water (lever arm x) is $0.434x^2 \times w dx$. Therefore the total torque action, about the chosen point, of the forces acting on the gate is equal to

$$0.434w \int_{x'}^{x''} x^2 dx = 0.434w \times \tfrac{1}{3}(x''^3 - x'^3)$$

whence, placing this equal to the torque action $XF[= X \times 0.434w(x''^2 - x'^2) \times \tfrac{1}{2}]$, we have equation (61).

113. Measurement of pressure. The barometer. The barometer consists of a glass tube T, Fig. 114, filled with mercury and inverted in an open vessel of mercury CC, the tube being of such length that an empty space V is left in which the pressure is zero.* The pressure in the tube at the level of the mercury in the open vessel is equal to atmospheric pressure, and it exceeds the pressure in the region V by the amount xdg according to equation (60). Therefore, since the pressure in V is zero, the

*Even if the tube is filled with extreme care so as to exclude all of the air, mercury vapor will form in the region V and the pressure will not be exactly zero.

value of atmospheric pressure is equal to xdg. This expression gives the value of atmospheric pressure in dynes per square centimeter, x being in centimeters, d being the density of the mercury in grams per cubic centimeter, and g being the acceleration of gravity in centimeters per second per second.

If the mercury is at some standard temperature, d is invariable; and if the barometer is used in a given locality, g is invariable; and *under these conditions the distance x may be used as a measure of the pressure*. In fact, atmospheric pressure is usually expressed in terms of the height the barometric column would have in millimeters or in inches if the mercury were at 0° C. and if the value of the acceleration of gravity were 981.61 cm./sec^2 (its value at 45° north latitude at sea level). To facilitate the accurate use of the barometer in different localities and at different temperatures, tables* have been published, with the help of which the height of barometric column under standard conditions as to temperature and gravity may be easily found from its observed height under known conditions.

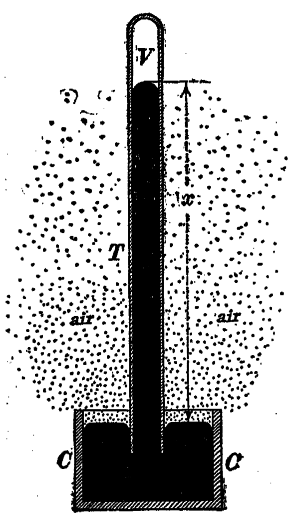


Fig. 114.

114. Measurement of pressure. Manometers or pressure gauges. The barometer is used for the measurement of the pressure of the atmosphere. An instrument for measuring the difference between the pressure in a closed vessel and atmospheric pressure is called a *manometer* or a *pressure gauge*.

The open tube manometer. When the pressure to be measured is small, for example, when it is desired to measure the

*To be found in many laboratory reference books. For example, in Kohlrausch's *Physical Measurements*, and in Landolt and Bornstein's *Physikalisch-Chemische Tabellen*.

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pressure of the gases at the base of a smoke-stack, or the pressure developed by a fan blower, the pressure is determined by measuring the height of water or mercury column which it will support. Thus Fig. 115 shows an open tube manometer arranged for measuring the pressure of the gas in city mains.

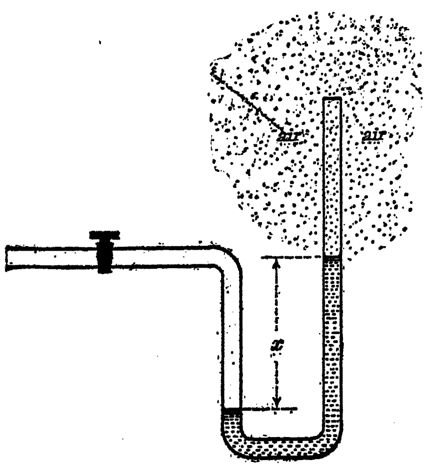


Fig. 115.

The Bourdon gauge. The pressure gauge commonly used on steam boilers is usually of the type known as the Bourdon gauge, of which the essential features are shown in Fig. 116. A very thin walled metal tube *abc* of flat elliptical section is closed at the end *c*, and the end *a* communicates through the tube *tt* with the steam boiler. The pressure inside of the tube *abc* tends to straighten it, and the movement of the end *c* actuates a pointer which plays over a scale the divisions of which are determined by calibration, that is, by noting the position of the pointer for various known pressures.

The gauge tester is a device for generating accurately known pressures which are communicated to a pressure gauge which is to be calibrated. It consists of a small metal chamber filled with oil. A plunger of known area *a* is forced into this chamber by a known weight, and the known pressure thus developed is communicated to the gauge.

115. Buoyant force of fluids. A fluid pushes upwards upon a body which is submerged in it and this upward force is called the buoyant force of the fluid. The buoyant force of a fluid upon a submerged body is equal to the weight of its volume of the fluid. This fact is called from its discoverer *Archimedes' principle.*

The point of application of the buoyant force is the center of figure* of the submerged body, and it is called the *center of buoyancy*.

The above statements may be made almost self-evident by the following considerations: Given a fluid at rest. *Imagine a certain portion of this fluid of any size and shape.* This *portion* is stationary, and therefore the surrounding fluid pushes upwards

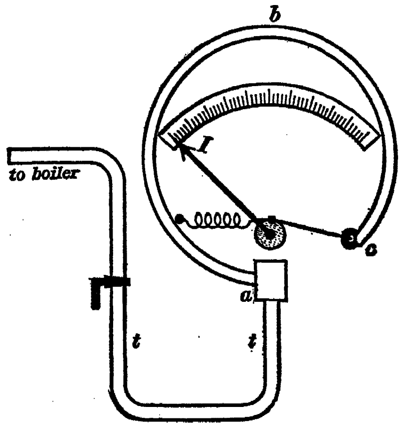


Fig. 116.

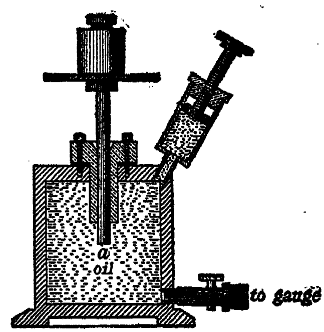


Fig. 117.

upon it with a force which is equal to its weight, and the point of application of this upward force is the center of mass of the portion. But the surrounding fluid acts upon the given portion of fluid in exactly the same way that it would act upon a submerged body of the same size and shape.

The principle of Archimedes is utilized in the ordinary method of finding the specific gravity of a body as follows: The body is weighed in the air and then it is suspended under water and weighed again. The difference is the weight of its volume of water, and the specific gravity of the body may then be calculated by dividing the weight (mass) of the body by the mass of its volume of water.

When a body is weighed on a balance scale its weight (mass) is underestimated if it is more bulky than the weights that are used

*Center of mass of the body if the body is homogeneous.

to balance it; this is on account of the greater buoyant force exerted by the air on the body than on the weights. This error is often quite appreciable, and it must be allowed for in accurate weighing.

A body which is partly submerged in a liquid is pushed upwards by a force which is equal to the weight of the displaced volume of liquid. Therefore *a floating body must displace its weight of the liquid in which it floats** *because the buoyant force must be just sufficient to balance the weight of the body.*

116. Equilibrium of a floating body.† A body is said to be in *unstable equilibrium* when the forces which act upon it tend to carry it farther and farther from its equilibrium position when it is displaced slightly therefrom. Thus a body standing vertically on a sharp point is in unstable equilibrium, the least displacement of the body in any direction causes it to fall over.

A body is said to be in *neutral equilibrium* when the forces which act upon the body remain in equilibrium as the body moves. Thus a homogeneous sphere resting on a smooth horizontal table, and a balanced wheel supported on an axle are in neutral equilibrium.

A body is said to be in *stable equilibrium* when the forces which act upon it tend to bring it back to its equilibrium position when it is displaced therefrom. Thus a weight fixed to the end of a spring, a pendulum hanging vertically downwards, and a block resting on a table are in stable equilibrium.

A body is said to have a high degree of stability when a very considerable force is required to displace it from its equilibrium position. Thus a broad sail-boat with its ballast placed low down in its hold is very stable, because a very considerable force is required to turn the boat from its vertical position.

Condition of equilibrium of a floating body. When a floating body is stationary, it is, of course, in equilibrium and the downward force of gravity must have the same line of action as the

*Effects of capillary action are here ignored.

†This subject is treated in detail in works on naval architecture.

upward force of buoyancy, otherwise these two forces would have an unbalanced torque action and the body would not be in equilibrium. Therefore *the center of a mass of a floating body and the center of figure of the submerged portion of the body (center of buoyancy) must lie in the same vertical line.*

The problem of determining the degree of stability of a floating body is greatly complicated by the change of shape of the submerged part of the body when the body is tilted to one side, and the shifting of the center of buoyancy which is due to this

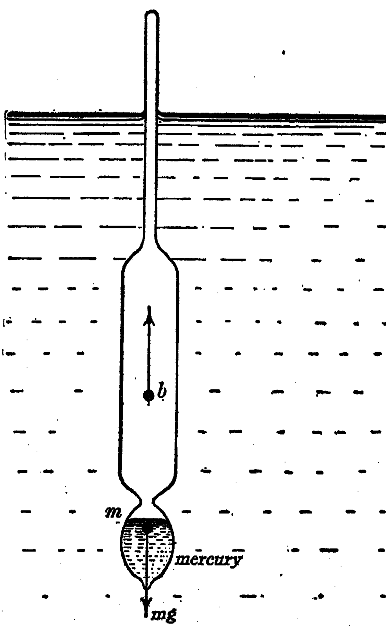


Fig. 118.

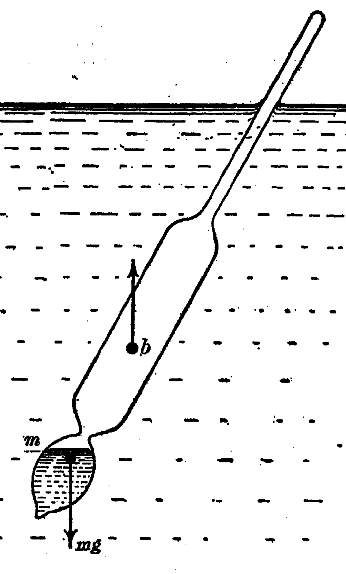


Fig. 119.

change of shape. Therefore the simplest case is that of a floating body of which the submerged portion does not change its shape when the body is tilted to one side.

Examples of simplest case. The submerged part of a floating sphere is the same in shape however the sphere be turned, and therefore the center of buoyancy does not move as the sphere is turned. If the center of mass of the sphere is at its geometrical

center we have a case of neutral equilibrium of floating; if the sphere is heavier on one side, it floats in stable equilibrium with its heavy side downwards, and in unstable equilibrium with its heavy side upwards.

The most interesting simple example of equilibrium of floating is the hydrometer as shown in Figs. 118 and 119. The shape

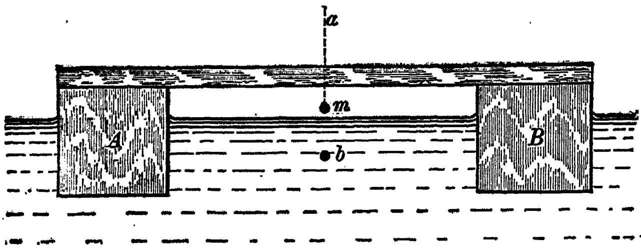


Fig. 120.

of the submerged portion is slightly altered when the hydrometer is tipped over, but the change of shape is nearly negligible and the center of buoyancy b is nearly fixed in position.

Example of the general case. Consider two floats A and B, Figs. 120 and 121, connected rigidly together by a beam. This arrangement is similar to the style of boat called a catamaran, and when it is in equilibrium the center of mass m and the center of buoyancy b are located as shown in Fig. 120. When, however, the arrangement is tilted, as shown in Fig. 121, the center of buoyancy b shifts towards the lower side, while the center of mass m of course remains stationary. *The*

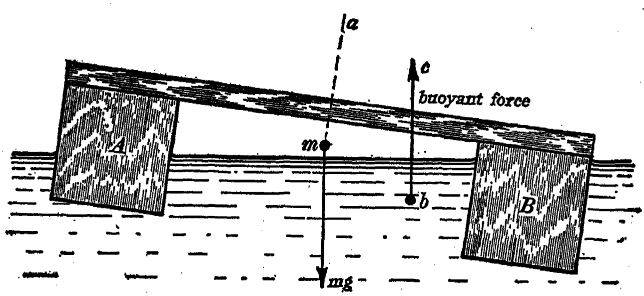


Fig. 121.

arrangement behaves, for slight angles of tilting, as if its center of buoyancy were fixed in the line ma at the place B where the line ma is cut by the vertical line bc in Fig. 121, *because the line of action of the buoyant force passes through the point B for any small angle of tilting.* The point B is called the *metacenter* of the float.

117. The hydrometer. The common form of the hydrometer is a light glass float, weighted at one end with lead or mercury, and having a cylindrical glass stem at the other end, as shown in Fig. 118. This float sinks to different depths in liquids of different specific gravities, and upon the stem is a scale which indicates the specific gravity of the liquid in which the instrument is placed.

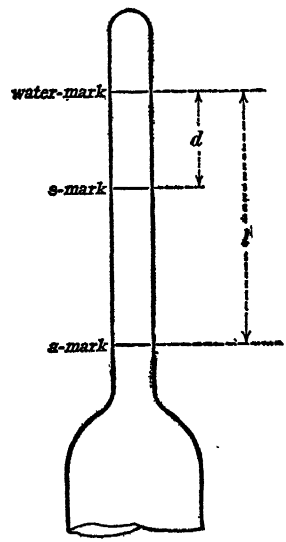


Fig. 122.

The specific gravity scale. To construct a specific gravity scale on the stem of a hydrometer, the instrument is floated in water and the water-mark located; the instrument is then floated in a liquid of known specific gravity a, and the a-mark is located; and then the distance l between the water-mark and the a-mark is measured. The scale is then determined by calculating the distance from the water-mark to each desired mark of the scale. Thus the distance d from the water-mark to the s-mark (s being a specific gravity, 1.10, 1.20, etc.) is given by the formula

$$d = l\left(\frac{1 - \frac{1}{s}}{1 - \frac{1}{a}}\right) \tag{62}$$

This equation may be derived as follows: A floating body displaces its weight of a liquid. The volume of water displaced by the instrument being taken as unity, the volume below the a-mark is $1/a$ and the volume below the s-mark is $1/s$ inasmuch as these liquids are a times as heavy and s times as heavy as water respectively. Therefore the volume of the length l of the stem is $(1 - 1/a)$ and the volume of the length d of the stem is

$(1 - 1/s)$, and, the stem being assumed cylindrical, the lengths l and d are proportional to these volumes.

Beaumé hydrometer scales. The specific gravity scale on a hydrometer is not a scale of equal parts and therefore the construction of the scale is tedious. On account of this fact a number of schemes have been proposed for constructing hydrometers with arbitrary scales of equal parts. Of these scales those of Beaumé are most extensively used.

Beaumé's scale for heavy liquids is constructed by locating the water-mark (near the top of the stem), and the mark to which the instrument sinks in a 15 per cent. solution of pure sodium chloride (common salt). The space between these marks is divided into 15 equal parts, and divisions of like size are continued down the stem. These divisions are numbered downwards from the water-mark. A liquid is said to have a specific gravity of 26° *Beaumé heavy* when the hydrometer sinks in it to mark number twenty-six on the scale here described.

Beaumé's scale for light liquids is constructed by locating the mark to which the instrument sinks in a 10 per cent. solution of sodium chloride (near the bottom of the stem), and the water-mark. The space between these marks is divided into 10 equal parts, and divisions of like size are continued up the stem. These divisions are numbered upwards from the salt solution mark. A liquid is said to have a specific gravity of 17° *Beaumé light* when the hydrometer sinks in it to mark number seventeen on the scale here described.

CAPILLARY PHENOMENA OF LIQUIDS.

118. Cohesion; adhesion. When a body is under stress, as for example a stretched wire, the tendency of the stress is to tear the contiguous parts of the body asunder. The forces which oppose this tendency and hold the contiguous parts of a body together are called the forces of *cohesion*. The forces which cause dissimilar substances to cling together are called the forces of *adhesion*. The discussion of the elastic properties of solids is a discussion of their properties of cohesion. The cohesion of water and the adhesion between water and glass are the forces which determine the curious behavior of water in a fine hair-like tube of glass, and the phenomena exhibited by liquids because of cohesion and adhesion are called *capillary phenomena* from the Latin word *capillaris* meaning hair.

119. Surface tension. On account of their cohesion, all liquids behave as if their free surfaces were stretched skins, that is, as if their free surfaces were under tension. Thus a drop of a liquid tends to assume a spherical shape on account of its surface tension. A mixture of water and alcohol may be made of the same density as olive oil, and a drop of olive oil suspended in such a mixture becomes perfectly spherical.

Many curious phenomena* are produced by the variation of the surface tension of a liquid with admixture of other liquids or with temperature. Thus a drop of kerosene spreads out in an ever widening layer on a clean water surface, on account of the fact that the tension of the clean water surface beyond the layer of oil is greater than the tension of the oily surface. A small shaving of camphor gum darts about in a very striking manner upon a clean water surface, on account of the fact that the camphor dissolves in the water more rapidly where the shaving happens to have a sharp projecting point, the water surface has a lessened tension where the camphor dissolves, and the greater tension on the opposite side pulls the shaving along. A thin layer of water on a horizontal glass plate draws itself away and leaves a dry spot where a drop of alcohol is let fall on the plate. A thin layer of lard on the bottom of a frying pan pulls itself away from the hotter parts of the pan and heaps itself up on the cooler parts, because of the greater surface tension of the cooler lard.

120. Angles of contact. Capillary elevation and depression. The clean surface of a liquid always meets the clean walls of a containing vessel at a definite angle. Thus a clean surface of water turns upwards and meets a clean glass wall tangentially, and a clean surface of mercury turns downwards and meets a clean glass wall at an angle of 51° 8′.

Since a clean water surface turns upwards and meets a glass wall tangentially it is evident that the surface of water in a small glass tube must be concave as shown in Fig. 123, and the result is that the water is drawn up into the tube. On

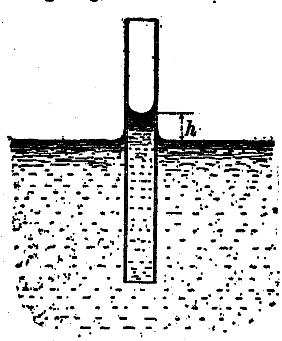


Fig. 123.

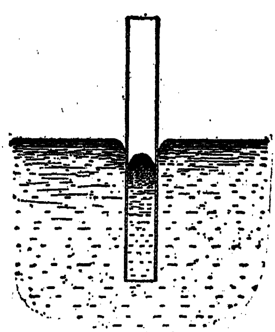


Fig. 124.

the other hand, the surface of mercury in a small glass tube is convex and the surface tension pulls the mercury down below the level of the surrounding mercury as shown in Fig. 124.

*See the very interesting article *capillary action* in the Encyclopedia Britannica. This article also gives a comprehensive discussion of the theory of capillary action.

121. Measurement of surface tension of water. Let r be the radius of the bore of the glass tube in Fig. 123. Then the circumference $2\pi r$ is the width of the surface film of water at the point of tangency, and $2\pi rT$ is the total upward force due to the tension of the film, T being the tension per unit width. The volume of water in the tube above the level of the surrounding water is πr^2h, and the weight of this water is πr^2hdg, where d is the density in grams per cubic centimeter and g is the acceleration of gravity. The weight of water in the tube being supported by the tension of the film, we have

$$2\pi rT = \pi r^2hdg$$

whence

$$T = \frac{rhdg}{2}$$

from which T may be calculated when r, d, and g are known and h observed. The surface tension of water is found in this way to be 81 dynes per centimeter breadth.

PROBLEMS.

145. Calculate the number of dynes per square centimeter in one pound-weight per square inch, taking the acceleration of gravity equal to 980 cm./sec^2. Ans. 68,900 dynes per square centimeter.

146. Figure 146*p* represents a hydrostatic press. The distances a and b are equal to 6 inches and 6 feet respectively, the

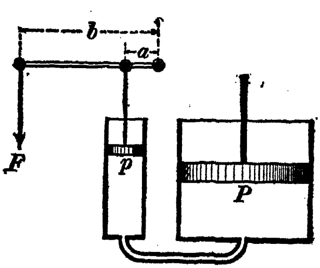


Fig. 146*p*.

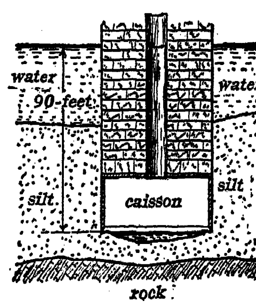


Fig. 149*p*.

diameter of the pump plunger p is 1.5 inches and the diameter of the press plunger P is 24 inches. Find the total force on P due to a force of 100 pounds at F, neglecting friction. Ans. 307,270 pounds-weight.

147. Calculate the circumferential tension and the longitudinal tension in the cylindrical shell of a boiler due to a steam pressure

of 125 pounds per square inch, the diameter of the boiler being 6 feet. Ans. 4,500 pounds-weight per inch and 2,250 pounds-weight per inch.

148. Sheet steel 0.02 inch thick will safely stand a tension of 200 pounds per inch of width. What is the greatest diameter of steel tube with 0.02 inch wall, which can safely withstand a pressure of 150 pounds per square inch? Ans. 2.66 inches.

149. Calculate the pressure of the air in the caisson shown in Fig. 149*p*, the distance from the water level in the river to the water level in the caisson being 90 feet. Ans. 39.1 pounds per square inch.

150. Oil and water are drawn up in two connecting tubes as shown in Fig. 150*p*. The height of the water column is 36 inches, and the height of the oil column is 42 inches. What is the

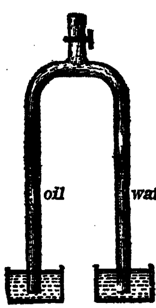


Fig. 150*p*.

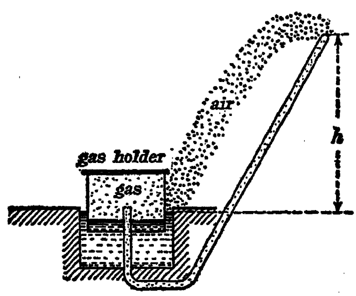


Fig. 151*p*.

ratio of the densities of oil and water (specific gravity of the oil)? Ans. 0.857.

Note. The difference in pressure between the air in the upper part of the tube in Fig. 150*p* and the outside air is equal to $x'd'g$ or to $x''d''g$, according to equation (60) where x' and x'' are the heights of the respective liquid columns and d' and d'' are the densities of the respective liquids.

151. The pressure of illuminating gas in a gas holder (at the base of the holder) exceeds the pressure of the outside air at the same level by an amount which is equivalent to 5.08 centimeters of water. Find the difference between the gas and air pressures on top of a hill at a height h of 12,200 centimeters above the gas

holder as shown in Fig. 151*p*. Ans. 9.956 grams-weight per square centimeter, or a pressure of 9.956 centimeters of water.

Note. The density of air under ordinary conditions is 0.0012 gram per cubic centimeter, the density of illuminating gas is, say, 0.0008 gram per cubic centimeter and the density of water is one gram per cubic centimeter. In solving this problem use equation (60) and if it is desired to express pressures in grams weight per square centimeter omit the factor g.

152. A vessel is filled with water and the hydrostatic pressure due to the water exerts a certain total force on the bottom of the vessel; sketch the form of the vessel for which the total force on the bottom of the vessel may be (*a*) greater than, (*b*) equal to, and (*c*) less than the weight of the contained liquid. Ans. (*a*) Sketch a vessel in the form of a truncated cone or pyramid standing on its large end; (*b*) sketch a vessel in the form of a cylinder or prism; (*c*) sketch a vessel in the form of a truncated cone or pyramid standing on its small end.

153. The density of mercury at 0° C. is 13.5956 grams per cubic centimeter. Calculate the value in dynes per square centimeter of standard atmospheric pressure, namely 76 cm. of mercury at 0° C., the value of gravity being 980.61 cm. per second per second. Give the result in dynes per square centimeter. Ans. 1,012,900 dynes per square centimeter.

154. The specific gravity of mercury is approximately 13.6. The pressure in pounds per square inch at a point x feet beneath pure water is $p = 0.434x$. Find the value in pounds per square inch of one English standard atmosphere, namely, 30 inches of mercury. Ans. 14.75 pounds per square inch.

155. Calculate the height of the *homogeneous atmosphere;* that is, assuming that the atmosphere has a uniform density of 0.00129 grams per cubic centimeter throughout, calculate the depth which would produce standard atmospheric pressure. Ans. 8,012 meters or 4.98 miles.

156. A piece of lead weighs 233.60 grams in air and 212.9 grams in water at 20° C. What is the specific gravity and the

density of lead at 20° C.? Ans. Specific gravity 11.285; density 11.265 grams per cubic centimeter.

Note. The density of water at 20°C. is 0.998252 gram per cubic centimeter.

157. A piece of glass weighs 260.7 grams in air and 153.8 grams in water at 20° C. The same piece of glass weighs 92.2 grams in dilute H_2SO_4 at 20° C. What is the specific gravity of the H_2SO_4 at 20° C.? Ans. 1.576.

158. A glass bulb weighs 75.405 grams when filled with air at standard temperature and pressure. It weighs 74.309 grams when the air is pumped out. It weighs 74.385 grams when filled with hydrogen at the same temperature and pressure. What is the specific gravity of hydrogen referred to air? Ans. 0.06926.

159. What is the net lifting capacity of a balloon containing 400 cubic meters of hydrogen, its material weighing 250 kilograms? (Weight of a cubic meter of air is 1,200 grams; weight of a cubic meter of hydrogen is 90 grams.) Ans. 194 kilograms.

160. The distance along a hydrometer stem from the water mark to the mark to which the instrument sinks in kerosene (specific gravity 0.79) is 9.62 centimeters. Calculate the distance from the water mark to the marks to which the instrument would sink in a 20 % solution of alcohol, in a 40 % solution of alcohol, in a 60 % solution of alcohol, in an 80 % solution of alcohol, and in pure alcohol. The specific gravities of these solutions are as follows: 20 % = 0.975; 40 % = 0.951; 60% = 0.913; 80 % = 0.863 100 % = 0.794. Ans. 0.93 centimeter, 1.86 centimeters, 3.44 centimeters, 5.75 centimeters, 9.39 centimeters.

161. The specific gravity of a 15 per cent. solution of sodium chloride at ordinary room temperature is 1.1115. Calculate the specific gravity corresponding to 26° Beaumé (heavy). Ans. 1.21.

Note. This problem is to be solved by using equation (62) in a way that will be apparent when it is considered that degrees Beaumé represent distances along the hydrometer stem.

162. The specific gravity of a 10 per cent. solution of sodium chloride at ordinary room temperature is 1.0734. Calculate the specific gravity corresponding to 20° Beaumé (light). Ans. 0.936.

163. Two rectangular boxes 12 inches × 12 inches × 16 feet are fastened to two cross-beams like a catamaran, and the whole weighs 625 pounds. The distance apart from center to center of the two boxes is 6 feet. Find how far to one side the center of buoyancy shifts when the raft is tilted 5° about its longitudinal axis, find the approximate position of the metacenter, and find the torque tending to bring the raft into a horizontal position. Ans. The center of buoyancy shifts 2.52 feet to one side; the metacenter is 28.9 feet above the center of mass of the float; and the torque tending to right the float is 1,575 pound-feet.

Note. For the sake of simplicity assume that the submerged part of each box remains rectangular. The metacenter is defined in terms of an infinitesimal angle of tilting, but its position for a 5° tilt may be determined approximately without the use of calculus in the case here considered.

CHAPTER IX.

HYDRAULICS.

[Throughout this chapter the following units are used: feet, feet per second, square feet, cubic feet, pounds (mass), pounds per cubic foot (density), pounds (force), pounds per square foot (pressure), and foot-pounds (energy). The factor g is equal to 32.2 which is the acceleration in feet per second per second of a one pound body when acted upon by an unbalanced force of one pound-weight.]

122. Limitations of this chapter. Hydraulics, in the general sense in which the term is here used, is the study of liquids and gases in motion; and the phenomena which are presented in this branch of physics are excessively complicated. Even the apparently steady flow of a great river through a smooth sandy channel is an endlessly intricate combination of boiling and whirling motion; and the jet of spray from a hydrant, or the burst of steam from the safety-valve of a locomotive, what is to be said of such things as these? Or let one consider the fitful motion of the wind as indicated by the swaying of trees and as actually visible in driven clouds of dust and smoke, or the sweep of the flames in a conflagration! These are *actual* examples of fluids in motion, and they are indescribably, infinitely* complicated. The finer details of such phenomena, however, are devoid of practical significance, indeed they present but little that is sufficiently definite even to be described.

*Everyone concedes the idea of infinity which is based upon abstract numerals (one, two, three, four and so on *ad infinitum*!), and the idea of infinity which is based on the notion of a straight line; but most men are wholly concerned with the humanly significant and persistent phases of the material world, their perception does penetrate into the substratum of utterly confused and erratic action which underlies every physical phenomenon, and they balk at the suggestion that the phenomena of fluid motion, for example, are infinitely complicated. Surely the abstract idea of infinity is as nothing compared with the intimation of infinity that comes from things that are seen and felt.

The science of hydraulics is based on ideas which refer to general aspects of fluid motion, like a sailor's idea of a ten-knot wind*; and, indeed, the engineer is concerned chiefly with what may be called *average effects* such as the time required to draw a pail of water from a hydrant, the loss of pressure in a line of pipe between a pump and a fire nozzle, or the force exerted by a water jet on the buckets of a water wheel. These are called average effects because they are never perfectly steady but always subject to perceptible fluctuations of an erratic character, and to think of any of these effects as having a definite value is, of course, to think of its average value under the given conditions.† The extent to which the practical science of hydraulics is limited is evident from the following outline of the ideal types of flow upon which nearly the whole of the science is based.

Permanent and varying states of flow. When a hydrant is suddenly opened, it takes an appreciable time for the flow of water to become steady. During this time (*a*) *the velocity at each point of the stream is increasing and perhaps changing in direction also.* After a short time, however, the flow becomes fully established and then (*b*) *the velocity at each point in the stream remains unchanged in magnitude and direction.*‡ The motion (*a*) is called a *varying state of flow*, and the motion (*b*) is called a *permanent state of flow.* Most of the following discussion applies to permanent states of flow, indeed there are but few cases in which it is important to consider varying states of flow.

The idea of simple flow. Stream lines. The idea of simple flow applies both to permanent and to varying states of flow, but it is sufficient to explain the idea in its application to permanent flow only. When water flows steadily through a pipe, the motion is always more or less complicated by continually changing eddies, the water at a given point *does not* continue to move in a

*See page 4.

†See two brief articles by W. S. Franklin, *Transactions of American Institute of Electrical Engineers*, Vol. XX, pages 285–286; and *Science*, Vol. XIV, pages 496–497, September 27, 1901.

‡Assuming the stream to be free from turbulence. See the following definition of simple flow.

fixed direction at a constant velocity; nevertheless, it is convenient to treat the motion as if the velocity of the water were in a fixed direction and of constant magnitude at each point. Such a motion is called a *simple flow*. In the case of a simple flow, a line

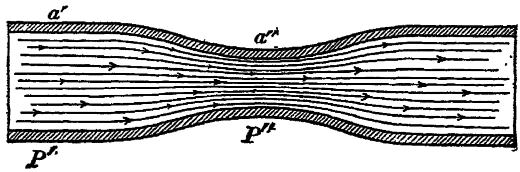


Fig. 125.

can be imagined to be drawn through the fluid so as to be at each point in the direction of the flow at that point. Such a line is called a *stream line*. Thus the fine lines in Fig. 125 are stream lines representing a simple flow of water through a contracted part of a pipe. To apply the idea of simple flow to an actual case of fluid motion is the same thing as to consider

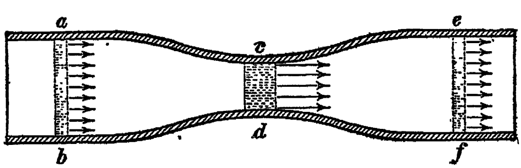


Fig. 126.

the *average character of the motion during a fairly long interval of time.*

Lamellar flow. Even though the motion of water in a pipe may be approximately a simple flow, the velocity may not be the same at every point in a given cross-section of the pipe, that is, the velocity may not be the same at every part of the layer *ab*, Fig. 126; in fact the water near the walls always moves slower than the water near the center of the pipe; nevertheless, it is convenient in many cases to treat the motion as if the velocity were the same at every point in any layer like *ab*, Fig. 126. Such an ideal flow is called a *lamellar flow*, because in such a flow the fluid in any layer or lamella *ab* would later be found in the layer

cd, and still later in the layer *ef*. To apply the idea of lamellar flow to an actual case of fluid motion is the same thing as to consider the *average velocity over the entire cross-section of a stream.*

Rotational and irrotational flow. In certain cases of fluid motion each particle of the fluid, if suddenly solidified, would be found to be rotating at a definite angular velocity about a definite axis; such fluid motion is called *rotational motion* or *vortex motion.* Thus the whirling motion of the water in an emptying sink is vortex motion. In other cases of fluid motion the particles of the fluid are not rotating; this kind of fluid motion is called *irrotational motion.* Some of the important practical aspects of vortex motion are discussed in the Encyclopedia Britannica article *Hydromechanics, Part III., Hydraulics,* section 30, 31, 103 and 190.

In irrotational fluid motion the velocity can be represented as a potential gradient, whereas in rotational fluid motion the velocity cannot be represented as a potential gradient. See Art. 18, space variation of vectors.

123. Some actual phenomena of fluid motion. The following treatment of fluid motion is so largely based upon the idea of simple lamellar flow that in pursuing the discussion we will be carried far away from any consideration of the details of actual

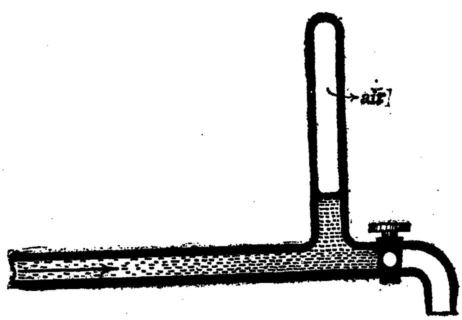


Fig. 127.

fluid motion, and, although many of these details are essentially erratic, still there are a few details which are definitely typical.

The water hammer. The most striking phenomenon that is associated with a varying state of fluid motion is the effect produced when an open hydrant is suddenly closed; the momentum of the water in the pipe causes the water to exert on the suddenly closed valve a momentary force very much like a hammer blow. This momentary force is often excessively large in value and a valve which is closed suddenly should be protected

by an air cushion as shown in Fig. 127. The sharp rattling noise which is occasionally produced in steam pipes is due to the "water hammer." A column of condensed water is driven along the pipe by the steam, the cooler steam ahead of the column condenses, and the column of water hammers against the end of the pipe or against a stationary body of water in the pipe.

The hydraulic ram consists of a valve A, Fig. 128, arranged to automatically open and close the end of a long pipe PP.

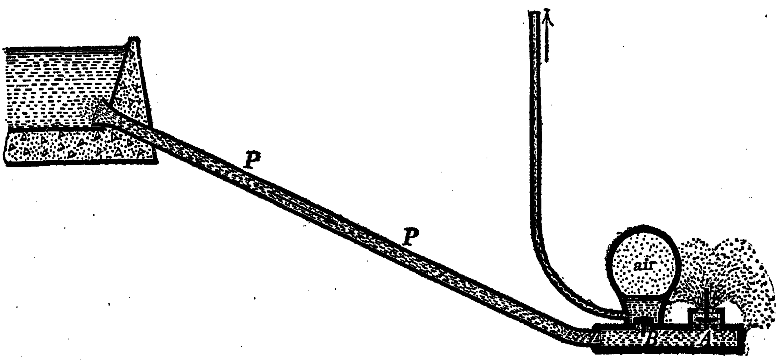


Fig. 128.

When the valve opens, the water from the dam starts to flow, this flow lifts the valve A thus suddenly closing the end of the pipe PP, and the momentum of the water in PP generates a momentary pressure which lifts the valve B and forces a small quantity of water to a high storage tank. The valve A then falls, and the action is repeated. The automatic opening of the valve A is due to the recoil of the water in the pipe PP as follows: At the moment when the water in PP is brought to rest in forcing water into the storage tank, the pressure at the end of PP is of course still excessive and the water near the end of PP is compressed. This compression then relieves itself by starting a momentary backward flow, or recoil, of the water in PP, and this recoil is followed by a momentary decrease of pressure sufficient to allow the valve A to drop.

The sensitive flame. When a fluid flows through a fairly

smooth walled pipe, the motion approximates very closely to a simple flow if the velocity is not excessive; but when the velocity is increased the motion tends to become more and more turbulent (full of eddies), and in many cases there is a fairly definite velocity at which the motion suddenly becomes very turbulent. This is shown by watching the movement of "sawdust-water" through a large glass tube. At low velocities the particles of sawdust move in fairly straight paths, but as the velocity of flow is increased the particles begin to gyrate with considerable violence when a certain critical velocity is reached.

This sudden increase of turbulence is illustrated by the familiar behavior of a gas flame. When the gas is turned on more and more the flame remains fairly steady until the velocity of the flowing gas reaches a certain critical value and then the flame suddenly becomes rough and unsteady. When the flame is on the verge of becoming unsteady it is sometimes very senitive; the least hissing noise causes it to become turbulent. An extremely sensitive flame may be obtained by burning ordinary illuminating gas from a smooth circular nozzle made by drawing a glass tube down to the desired size (about ½ millimeter to 1 millimeter diameter of opening). Generally, several nozzles must be tried before one is found that is suited to the gas pressure that is available.

Vortex rings. When a fluid is at rest, mixing takes place only by the very slow process of diffusion, and when a fluid is in turbulent motion the mixing of the different parts of the fluid takes place very rapidly on account of the eddies which constitute the turbulence. The slowness of mixing of a smoothly flowing fluid, however, is illustrated by the smooth gas flame and by the threads of smoke that rise from the end of a cigar. Such a stream of fluid flowing smoothly through a large body of fluid at rest tends always to break up into what are called vortex rings. Thus a fine jet of colored water entering at the top of a large vessel of clear water and streaming towards the bottom, breaks up into rings which spread out wider and wider as they move

downwards, each ring preserving its identity (not mixing with the clear water). The most interesting example of the formation of vortex rings is the familiar case of the formation of smoke rings when smoke issues as a moderate puff from an orifice into the air. Of course the smoke only serves to make the rings visible, and a candle can be blown out by an invisible vortex ring of air projected across a large room from an orifice in a box by striking a flexible diaphragm which is stretched like a drum head across the back of the box.

Cyclonic movements. When water flows out of a hole in the bottom of a bowl a whirlpool generally forms above the hole. The formation of this whirlpool depends upon the existence of a slow rotatory motion of the water in the bowl, which rotatory motion is greatly increased when the water flows toward the hole as a center. This increase of velocity as a rotating particle is made to approach the center may be explained with the help of Fig. 129*a* as follows: A ball B is twirled on a string s which passes

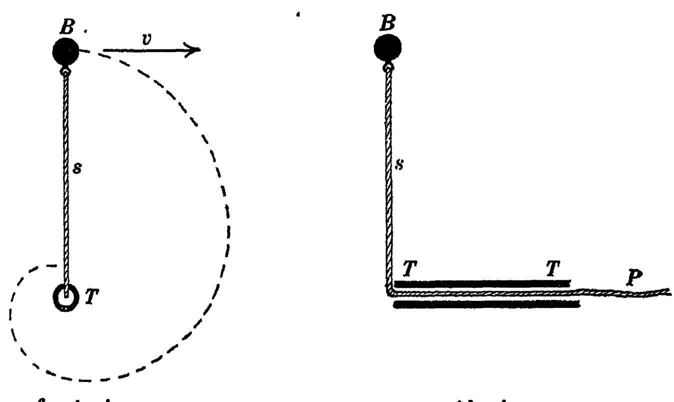


Fig. 129*a*.

through a tube TT, and after the ball is set in motion the string is pulled by taking hold of it at P, thus causing the ball to move along the dotted spiral. The velocity of the ball is increased as it moves towards the axis; the work done in pulling the string increases the kinetic energy of the ball.

Figure 129*b* represents a whirlpool of water in a deep circular bowl. In this case the values of the hydrostatic pressure at the points *a*, *b* and *c* are proportional to the lengths of the respective dotted lines, that is to say, the hydrostatic pressure decreases towards the axis of the bowl, and it is this decrease of hydrostatic pressure which causes an increase of velocity of a given particle of water as it moves towards the axis of the bowl,* as explained in Arts. 126, 127 and 128.

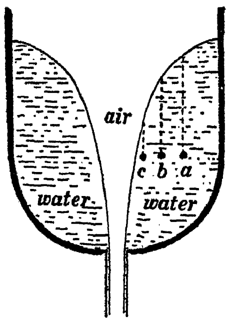


Fig. 129*b*.

The rotation of the earth on its axis involves a slow motion of turning of one's horizon about a vertical axis (except at the equator). When the warm air near the earth's surface starts to flow upwards at a given point, a chimney-like effect is produced by the rising column of warm air, the lower layers of warm air flow towards this "chimney" from all sides, and the slow turning motion of the horizon becomes very greatly exaggerated in a more or less violent whirl at the "chimney" which is the center of the storm. The *cyclone* is a storm movement of this kind covering hundreds of thousands of square miles of territory with a central chimney hundreds of miles in diameter, the *tornado* is a storm movement of this kind covering only a few square miles of territory with a central chimney seldom more than a thousand yards in diameter. The whirling motion near the center of a tornado is often excessively violent.

124. Rate of discharge of a stream. The volume of water which is delivered per second by a stream is called the *discharge rate* of the stream. Thus the mean discharge rate of the Niagara River is 300,000 cubic feet per second. *The rate of discharge of a stream is equal to the product of the avergae velocity, v, of the stream*

*This increase of rotatory motion due to the movement of a portion of a rotating system towards the axis of rotation is an example of the constancy of spin momentum of a system on which no outside force (torque) acts, as explained in Art. 62.

and the sectional area, a, of the stream. For example, let PP, Fig. 130, be the end of a pipe out of which water is flowing, and let us assume that the velocity of flow has the same value v over the entire section of the stream (lamellar flow), then the water which flows out of the end of the pipe in t seconds would make a cylinder or prism of length vt, and of sectional area a, as indicated in the figure, and the volume of this water is therefore avt. Dividing this volume by the time t gives the discharge rate av.

Variation of velocity with sectional area of a steady stream. Consider a simple flow of water through a pipe as indicated by the stream lines in Fig. 125. Let a' and a'' be the cross-sectional areas of the stream at any two points P' and P'' and let v', and

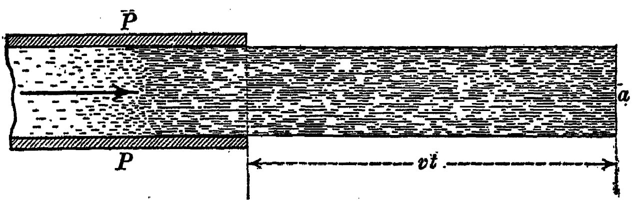


Fig. 130.

v'' be the average velocities of the stream at P' and P'' respectively. Then $a'v'$ is the volume of water which passes the point P' per second, and $a''v''$ is the volume of water which passes the point P'' per second; and, therefore, since the same amount of water must pass each point per second, we have

$$a'v' = a''v'' \tag{63}$$

that is, the product av has the same value all along the pipe, so that v is large where a is small, and v is small where a is large.

Equation (63) applies only to a fluid which is approximately incompressible like water or any other liquid. In such a case $a'v'$ is the amount of water per second entering one end of a pipe and $a''v''$ is the amount of water per second flowing out of the other end of the pipe, and these two expressions must be equal to each other. If, however, the fluid is compressible like a gas, then equation (63) becomes

$$a'v'd' = a''v''d'' \tag{64}$$

where a' is the sectional area of the steady stream of gas at one place, v' is the average velocity of the stream at that place, d' is the density of the gas at that place, and a'', v'' and d'' are the cross sectional area, the velocity of the stream and the density of the gas at another part of the stream.

125. The ideal frictionless incompressible fluid. When a jet of water issues from a tank, there is a certain relation between the velocity of the jet and the difference in pressure inside and outside of the tank. When there are variations of the velocity of flow of water along a pipe due to enlargements or contractions of the pipe [see equation (63)], the pressure decreases wherever the velocity increases and *vice versa*. These mutually dependent changes of velocity and pressure are always complicated by friction, and by the variations of the density of the fluid due to the variations of pressure; and in order to gain the simplest possible idea of these mutually dependent changes of velocity and pressure the conception of the *frictionless incompressible fluid* is very useful.

When the water in a pail is set in motion by stirring, it soon comes to rest when left to itself. A fluid which would continue to move indefinitely after stirring would be called a frictionless fluid.

When a moving fluid is brought to rest by friction, the kinetic energy of the moving fluid is converted into heat and lost. Such a loss of energy would not take place in a frictionless fluid, and therefore the total energy (kinetic energy plus potential energy) of a frictionless fluid would be constant. *This principle of the constancy of total energy is the basis of the following discussion of the flow of the ideal frictionless fluid.* The following discussion applies to fluids which are not only frictionless but also incompressible. In fact, ordinary liquids are nearly incompressible.

126. Energy of a liquid. (*a*) *Potential energy per unit of volume.* Work must be done to pump a liquid into a region under pressure, the amount of work done in pumping one unit of volume of the liquid is the potential energy per unit of volume of the

liquid in the high pressure region, and it is equal to the pressure. That is

$$W' = p \tag{65}$$

In this equation W' and p may both be expressed in c.g.s. units or in the units enumerated at the head of this chapter.

Proof of equation (65). Let CC, Fig. 131, be the cylinder of a pump which is used to pump liquid into a tank under a pressure of p pounds per square foot, and let the area of the piston be a square feet. Then the force required to move the piston (ignoring friction) is ap pounds, and the work done in moving the piston through a distance of l feet is apl foot-pounds. But al is the volume of water pushed into the tank

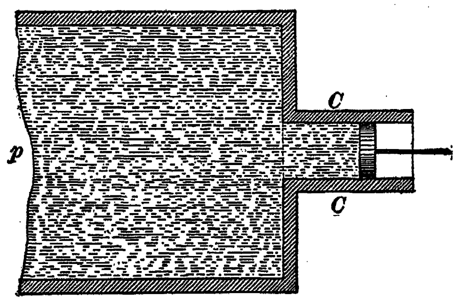


Fig. 131.

by the movement of the piston, and therefore, dividing apl by al gives the work in foot-pounds required to push one cubic foot of water into the tank.

When a stream of liquid moves in a horizontal plane, the gravity pull of the earth does no work on the liquid; but when a stream flows through an inclined pipe the gravity pull of the earth does work (positively or negatively) on the liquid and it is necessary therefore in this case to consider the energy of altitude as a part of the potential energy of the liquid. In fact, one cubic foot of liquid of which the weight is d pounds has potential energy equal to hd foot-pounds when it is at a height of h feet above a chosen reference level, so that the potential energy of the liquid per cubic foot is

$$W' = p + hd \tag{66}$$

In the discussion in Arts. 128 and 129 the effects of gravity are ignored, that is to say, the pipe or stream is supposed to be horizontal. If it is desired to consider the effects of gravity, $p + hd$ can be substituted for p in the equations of Art. 129.

(*b*) *Kinetic energy.* Let v be the velocity in feet per second of a moving liquid, and let d be the mass of one cubic foot of the liquid in pounds (d is the density of the liquid). Then the kinetic energy of one cubic foot of the liquid in foot-pounds is

$$W'' = \frac{1}{2g} dv^2 \tag{67}$$

according to equation (27) of chapter V.

127. Efflux of a liquid from a tank. Consider a tank, Fig. 132, containing a liquid of which the density is d pounds per cubic foot. Let oo be an orifice from which the liquid issues as a jet at a velocity v feet per second to be determined. Let p pounds per square foot be the pressure in the tank at the level of the orifice, and let p' be the outside pressure (atmospheric pressure). In the tank, where the velocity of the liquid is inappreciable, the total energy of the liquid per unit of volume is the potential energy p [equation (65)]. In the jet the total energy per unit volume is $p' + 1/2g \times dv^2$ [equations (65) and (67)]. As a portion of the liquid moves from the tank into the jet its total energy would remain unchanged if it were frictionless, so that we would have

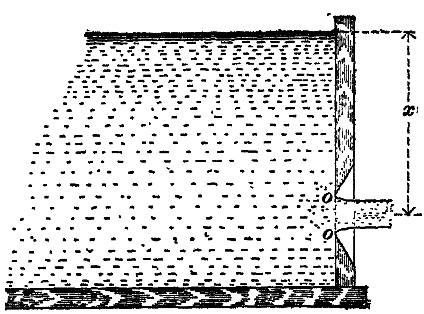


Fig. 132.

$$p = p' + \frac{1}{2g} dv^2$$

whence

$$v = \sqrt{\frac{2g(p - p')}{d}} \tag{68}$$

This equation expresses the velocity of efflux of a frictionless incompressible fluid. The effect of friction is to decrease v, and the effect of compressibility is to increase v. For ordinary liquids the effect of friction is the greater, and equation (68) gives too large a value for v. For gases the effect of compressibility is the greater, and equation (68) gives too small a value for v.

Torricelli's theroem. The velocity of efflux of a frictionless liquid is equal to the velocity a body would gain in falling freely through the distance x of Fig. 132. The pressure-difference $p - p'$ is equal to xd, according to equation (60)b of chapter VIII, so that, substituting xd for $p - p'$ in equation (68), we have

$$v = \sqrt{2gx}$$

and this is the velocity gained by a body in falling through the distance x, according to Art. 35.

The kinetic energy, per cubic foot, of the liquid in the jet is equal to the work that would be done by gravity on one cubic foot of water in falling from the surface of the liquid in the tank to the level of the orifice.

128. Bernoulli's principle. Consider a stream of water flowing through a pipe. Wherever the pipe is small in section the velocity of the water is great, and wherever the pipe is large in section the velocity of the water is small, according to equation (63).

Let p be the pressure of the water at a point in a pipe, and v the velocity of the water at that point. Then the potential energy of the water per cubic foot is p, and the kinetic energy of the water per cubic foot is $\frac{1}{2g}dv^2$ according to Art. 126, and if the liquid is frictionless so that no energy can be lost (converted into heat), then the total energy $p + \frac{1}{2g}dv^2$ must be constant everywhere along the stream. Therefore, where the velocity is great, the pressure must be small, and where the velocity is small, the

pressure must be great. This relation was first established by John Bernoulli and it is known as *Bernoulli's principle.*

Examples. (*a*) *The disk paradox.* Figure 133 represents a short piece of tube T ending in a flat disk DD, and dd is a light metal disk which is prevented from moving sidewise by a pin which projects into the end of the tube T. If one blows hard through the tube T, the disk dd is held tight against DD because of the low pressure in the very greatly contracted portion of the air stream between the disks. In fact, the pressure of the air in the region between the disks is less than atmospheric pressure, and it increases towards the edge of the disks as the velocity of the air stream diminishes (and the sectional area of the stream increases).

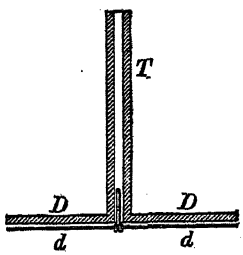


Fig. 133.

(*b*) *The jet pump.* The essential features of the jet jump are shown in Fig. 134. Water from a fairly high-pressure supply H

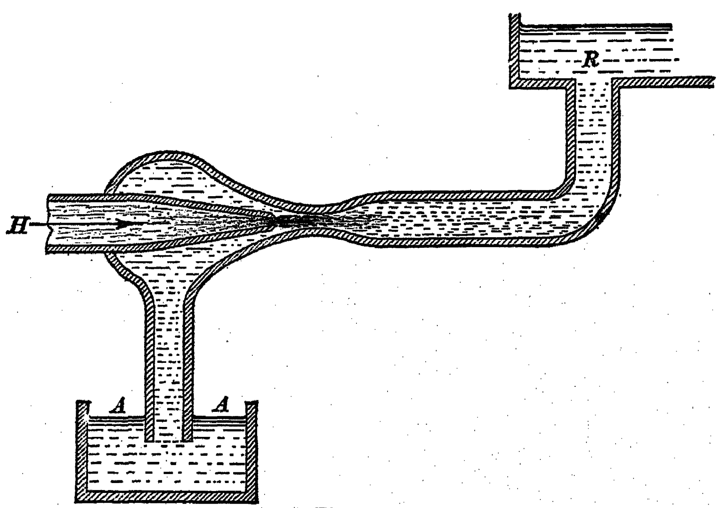


Fig. 134.

nters a narrow throat, the low pressure in the throat sucks water rom AA, and the water from H, together with the water from

AA, is discharged into the reservoir R. This type of pump is frequently used for pumping water out of cellars, and it is extensively used as an air pump in chemical laboratories.

The steam injector is a jet pump, and its paradoxical action in pumping water into a boiler at the same pressure* as the steam supply depends upon the low density of the steam and upon the fact that the steam is condensed in the injector. The low-density steam acquires a very high velocity in flowing out of the boiler, according to equation (68), and the velocity which is imparted to the water in the injector (including the water from the condensed steam) is sufficient to carry the water back into the boiler.

(*c*) The volume of water discharged per second from a given sized orifice oo, Fig. 135, is greatly increased by the flaring tube

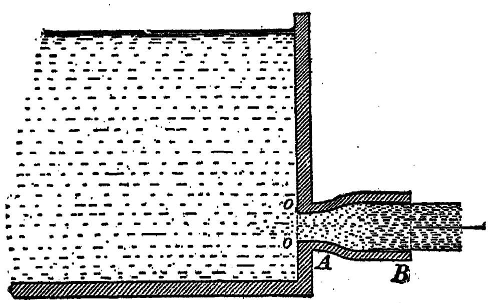


Fig. 135.

AB. The rate of discharge of a frictionless fluid would depend only upon the size of the open end B of the tube, the contraction at A would have no effect. In the case of an actual fluid, the effect of the contraction at A is to increase the friction considerably and thus reduce the discharge rate below what it would be if the tube at A were as large as at B.

(*d*) *The curved flight of a base-ball.* Figure 136*a* shows a stream of air flowing past a stationary ball. In this case the velocity of the air is the same at the points a and b, and therefore, the pressure

*Indeed the injector can pump water into a region in which the pressure is greater than the pressure of the supply stream.

of the air is the same on both sides of the ball. If, however, the ball is rotating and if its surface is slightly rough, the surrounding air will be set into a whirl, and in this case the stream lines of the

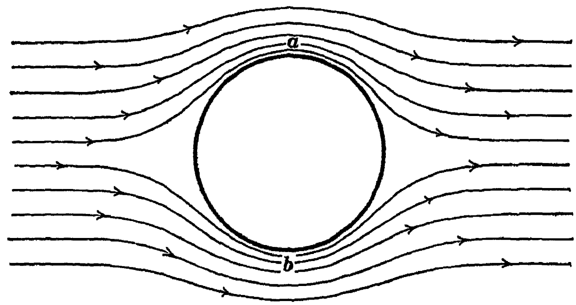


Fig. 136*a*.

air flowing past the ball will be some what as shown in Fig. 136*b*. The velocity at each point in this figure is the resultant of two velocities, namely, the velocity due to the stream alone (as shown in Fig. 136*a*), and the velocity due to the whirl alone. In Fig. 136*b* the velocity of the air is greater in the region *b* than it is in the region *a*, and according to Bernoulli's principle the pressure

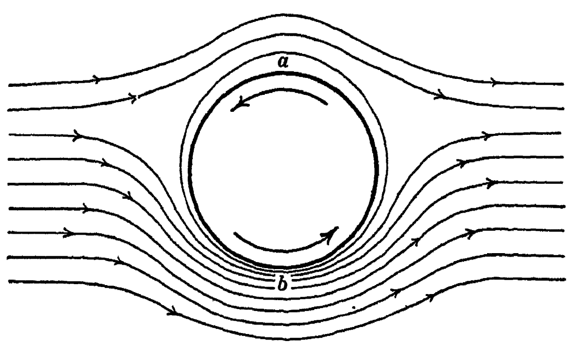


Fig. 136*b*.

of the air is less in the region *b* than it is in the region *a*. Therefore the excess of pressure in the region *a* pushes the ball from *a* towards *b*.

Figures 136*a* and 136*b* show a stream of air flowing past a stationary ball. The same effects, however, are produced when the

ball moves through a stationary body of air in the direction from right to left in either figure.

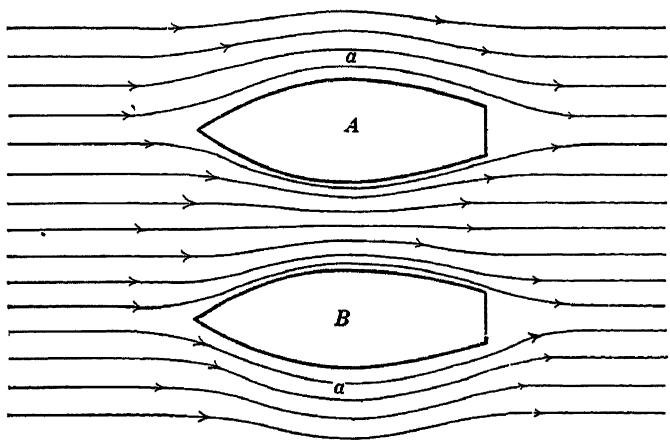


Fig. 137.

(*e*) An interesting phenomenon dependent upon Bernoulli's principle is that two ships steaming along side by side attract each other and are in danger of being drawn together. Thus

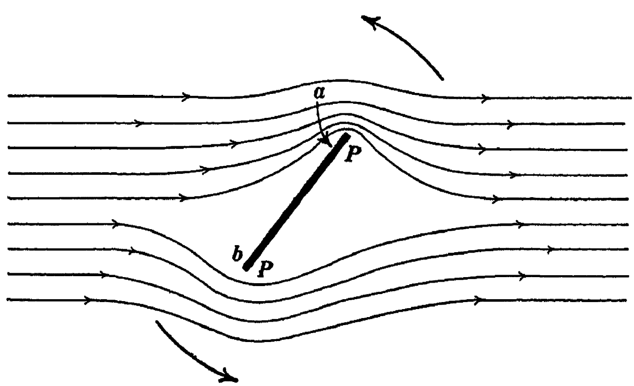


Fig. 138.

when a squadron of warships is maneuvering, it is dangerous for one ship to attempt to pass close by another. Figure 137 represents two stationary boats *A* and *B* with a stream of water flow-

ing past them. The velocity of the water is much greater in the region between the ships than it is on either side as indicated by the stream lines, therefore the pressure in the region between the ships is less* than the pressure in the regions aa, and therefore the ships are pushed towards each other. Figure 137 represents a stream of water flowing past two stationary ships; the effect, however, is the same if the two ships are moving through a large body of still water.

(*f*) Figure 138 shows the approximate trend of the lines of flow of a stream in the neighborhood of a flat plate PP. The velocity of the stream is very great at the point a and small at the point b. Therefore the pressure of the fluid is great at b and small at a, and this difference in pressure tends to cause the plate to rotate in the direction of the curved arrows, thus bringing the plate into a position with its plane at right angles to the stream.

129. Diminution of pressure in a throat. A contracted portion of a pipe is called a *throat*. When a fluid flows through a

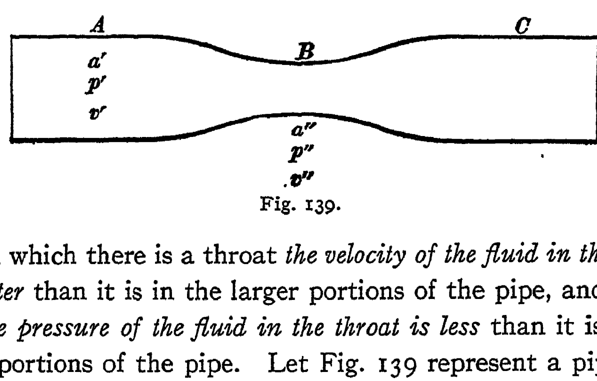


Fig. 139.

pipe in which there is a throat *the velocity of the fluid in the throat is greater* than it is in the larger portions of the pipe, and therefore *the pressure of the fluid in the throat is less* than it is in the larger portions of the pipe. Let Fig. 139 represent a pipe with a throat; let a' be the cross-sectional area of the pipe at A, and let p' and v' be the pressure and velocity respectively of the fluid at A; and let a'', p'' and v'' be the corresponding quantities at B. Then, if the fluid is incompressible, we have

$$a'v' = a''v'' \qquad \text{(i)}$$

*The water level between the two boats is lower than the normal water level.

according to equation (63), and if the fluid is also frictionless we have

$$p' + \frac{1}{2g} dv'^2 = p'' + \frac{1}{2g} dv''^2 \tag{ii}$$

Therefore, substituting the value of v'' from (i) in (ii) and solving for $p' - p''$, we have

$$p' - p'' = \frac{1}{2g}\left(\frac{a'^2 - a''^2}{a''^2}\right) dv'^2 \tag{69}$$

where $p' - p''$ is the diminution of pressure in the throat.

The diminution of pressure in a throat is explained directly from Newton's second law of motion as follows: Consider a particle of liquid at A, Fig. 139. This particle gains velocity as it approaches B, and loses velocity again as it approaches C. Therefore, an unbalanced force must be pushing the particle forwards as it passes from A to B, that is, the pressure in the region behind the particle is greater than the pressure in the region ahead of the particle; and an unbalanced force must be opposing the motion of the particle as it passes from B to C, that is, the pressure ahead of the particle is greater than the pressure behind it.

The diminution of pressure at a throat in a pipe is, of course, an example of Bernoulli's principle, and it is exemplified by the disk paradox which is described in Art. 128.

The Venturi water meter consists of a throat inserted in a water pipe through which the water to be measured flows. The diminution of pressure $p' - p''$ [see equation (69)] is measured, and, since the cross-sectional areas a' and a'' are known, the velocity v' and the rate of discharge $a'v'$ can be calculated from the measured value of $p' - p''$.

130. Reaction of a water jet. Force of impact of a jet. Figure 140 represents a tank containing water at a pressure p (in excess of outside pressure). The tank has an orifice of area a, and the orifice is closed by a plug P. The force acting on the plug is equal to pa, and the total force pushing on the side AA of the

tank is equal to total force pushing on the side BB *including the force acting on the plug.* Therefore, it would seem that an unbalanced force equal to pa would push the tank towards the left in Fig. 140 if the plug were removed; but when the plug is removed there is a reduction of pressure in the neighborhood of the orifice as indicated by the very small arrows in Fig. 141, so that the unbalanced force which pushes the tank towards the left in Fig. 141 is much greater than pa, it is in fact equal to

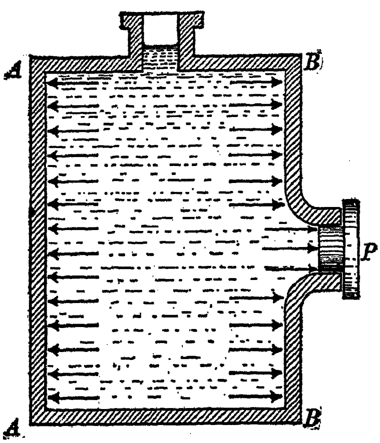


Fig. 140.

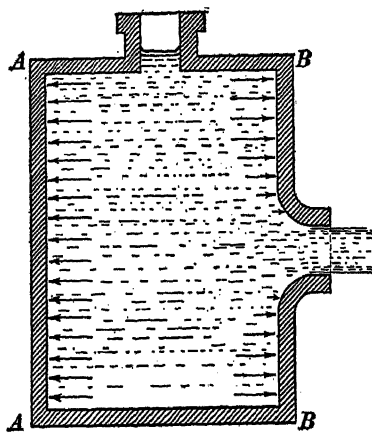


Fig. 141.

$2pa$ on the following assumptions, namely, (*a*) that the velocity of efflux is that of an ideal incompressible fluid, and (*b*) that the jet issues as a parallel stream of the same size as the orifice.

It is difficult, however, to show that the reaction of the jet is $2pa$ by considering the change of pressure inside of the tank due to the existence of the jet, but the reaction can be evaluated in a comparatively simple manner by considering the force which must act on the outflowing water to set it in motion. In one second av cubic feet or avd pounds of water flow out of the orifice, and this amount of water has gained velocity v. To impart velocity v to avd pounds in one second requires a force equal to $adv \times v \div g$ pounds-weight, according to equation (5), and, of course, the jet must push backwards upon the tank with

n equal force. Therefore the reaction of the jet is adv^2/g ounds-weight; but the velocity of efflux and difference of presure $p[= p' - p''$ of equation (68)] satisfy the equation

$$\frac{1}{2g} dv^2 = p$$

o that

$$\frac{1}{g} adv^2 = 2pa.$$

When a jet of water strikes an obstacle so as to be brought to est, it exerts a force equal to adv^2/g on the obstacle. This is vident when we consider that the force which is exerted by . jet as the water is brought to rest is equal to the force which nust be exerted on the water to set it in motion at the point vhere the jet is produced. If the jet strikes a flat plate so as to ebound in a direction at right angles to its original velocity, s indicated in Fig. 142, then it exerts the same force as it would

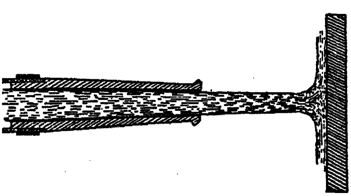


Fig. 142.

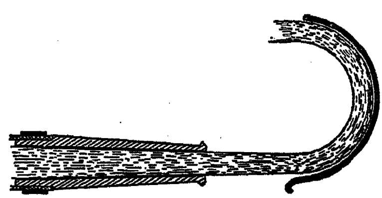


Fig. 143.

xert if it were brought to rest, because it loses all of its velocity n the original direction. If the jet strikes a curved plate as ndicated in Fig. 143 so as to rebound in an opposite direction vith unchanged velocity (gliding along the curved plate without riction), then it would exert twice as much force as it would xert if it were brought to rest, because it loses its original veloc- ty and gains an equal amount in the opposite direction.

131. The Pitot tube. A glass tube drawn to a moderately fine oint is placed in a stream of water moving at velocity v, as hown in Fig. 144. In accordance with what is stated above oncerning the reaction and impact of a jet, the water in the tube

must stand above the level of the stream at a height h which is approximately *twice* as great as the height which would cause an efflux velocity equal to v. That is, the velocity of the stream is approximately equal to $\sqrt{gh}$, according to Art. 127. This device is called the *Pitot tube*, it is frequently used for measuring the velocity of streams,* and, when so used, it is usually arranged

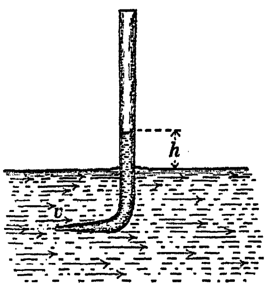


Fig. 144.

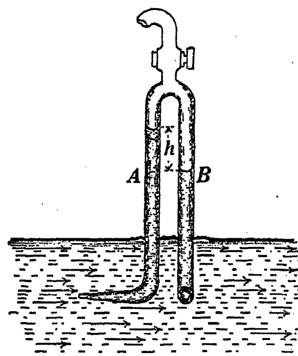


Fig. 145.

as shown in Fig. 145, so as to bring the difference of level h into a convenient position for measurement. The tube A, Fig. 145, has its point directed against the stream, and the tube B has its point directed at right angles to the stream. By drawing the device, Fig. 145, through still water at a known velocity, or by using it to measure a velocity which has been determined by other means, it has been found that its indications are accurate to about one per cent. when the tubes are drawn to small points, as shown in Fig. 145.

132. Gauging of streams. To gauge a stream is to determine the volume of water discharged by the stream per second. This determination depends upon the measurement of the sectional area a of the stream and of the mean velocity v of the stream, and the discharge rate of the stream is equal to av according to equation (63).

*Other methods for measuring the velocity of a stream are often used in practice. See, for example, Merriman's *Hydraulics*.

Small streams are usually gauged by means of an orifice in a temporary dam.* Let x be the distance of the center of the orifice beneath the surface of the water in the dam, then the velocity of efflux would be equal to $\sqrt{2gx}$ if the water were frictionless, and the product of this velocity and the area of the orifice a would be the discharge rate if the flow in the orifice were lamellar. Experiments show that the mean velocity of a water jet flowing from a sharp edged orifice like that shown in Fig. 132 is about 0.98 of the value ($\sqrt{2gx}$) corresponding to ideal frictionless flow; and experiment shows that the cross-sectional area of the jet at a short distance from the orifice (where the flow becomes approximately lamellar) is about 0.62 of the area of the orifice, provided the orifice has sharp edges and is in the middle of a flat wall. Therefore the rate of discharge from an orifice like that shown in Fig. 132 is approximately equal to $0.98 \times 0.62 \times a\sqrt{2gx}$.

A large river is gauged by determining the cross-sectional area of the river and measuring the velocity of the water at a large number of points in the section so as to determine the average velocity. The velocity of the current is sometimes measured by means of floats, sometimes by means of Pitot tubes, and sometimes by means of a so-called current meter which consists of a rotating wheel like a screw propeller which drives a speed counting device. The current meter has to be calibrated by observing its speeds when it is dragged through still water at various known velocities.

133. Fluid friction. The dragging forces which oppose the motion of a body through the air or water, and the dragging forces which oppose the flow of fluids through pipes and channels are due to a type of friction which is called *fluid friction.*

Friction of fluids in pipes and channels. There are two fairly distinct actions which are involved in the friction of fluids in pipes and channels, and, although these two actions always exist to-

*The arrangement called a weir is a notch in the top of a temporary dam, and the formulas for calculating the discharge rate over a weir may be found in any treatise on Hydraulics.

gether, it is instructive to consider two extreme cases in which the two actions are approximately separated.

Viscous Friction. When a fluid flows through a very small, smooth-bore pipe, the loss of pressure is proportional to the rate of discharge, or to the mean velocity of flow of the fluid in the pipe. This fact was first established by Poiseuille (1843). In fact, for this case we have

$$p = \frac{8\eta l Q}{\pi R^4} \tag{70}$$

in which l is the length of the tube in feet, R is the radius of its bore in fractions of a foot, Q is the volume of liquid in fractions of a cubic foot discharged per second, and η is a constant called the *coefficient of viscosity* of the liquid. It is evident from this equation that the loss of pressure due to viscous friction is very small indeed when the radius R of the tube is moderately large. In fact, viscous friction is nearly always negligible under practical conditions. A full discussion of equation (70) and a definition of the coefficient of viscosity are given in Arts. 134 and 135.

*Eddy Friction.** Consider a series of chambers, $ABCD$, Fig. 146, communicating with each other through narrow orifices, and

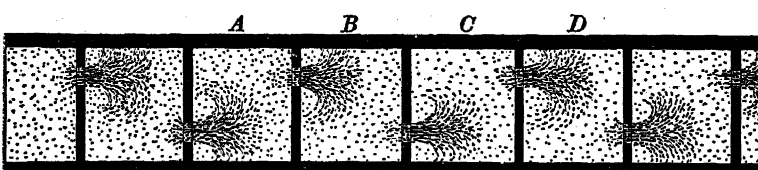


Fig. 146.

let us suppose water to flow through this series of chambers. As the water enters an orifice it gains a certain amount of velocity v, and decreases in pressure by the amount $1/2g \times dv^2$, according to Art. 127. The velocity so gained is lost by eddy action in the next chamber, and when the water flows through the next orifice it must gain velocity anew and suffer a corresponding

*A fluid entirely devoid of viscosity would not form eddies, so that all fluid friction is due to viscosity, directly or indirectly.

drop in pressure, as before. It is therefore evident that *the drop of pressure through the series of chambers, ABCD, is proportional to the square of the rate of discharge.* This law of eddy friction is verified by experiment for a series of chambers as shown in Fig. 146, where the eddies are definitely *localized.* In an ordinary pipe, however, there is a tendency for the eddy movements to become finer grained, as it were, with increasing velocity; that is, with increased velocity, a given particle of fluid acquires velocity and loses it again an increased number of times in traveling a given distance. The consequence of this fact is that the loss of pressure due to eddy friction increases more rapidly than in proportion to the square of the rate of discharge.

In all ordinary cases of the flow of fluids through pipes and channels, eddy friction is very much larger than viscous friction, and the practical formula for calculating the loss of pressure due to the flow of a fluid through a given length of pipe of a given size is based upon the assumption that the loss of pressure is proportional to the density of the fluid and to the square of the rate of discharge, or indeed, to the square of the velocity of the fluid if the pipe is of uniform size.

Practical formula for calculating the frictional loss of pressure due to the flow of water or gas through a pipe. The formula which is used in practice for calculating the frictional loss of pressure in a pipe is only approximately true and therefore the formula has no rigorous derivation. The only thing to be done in connection with it is to exhibit its meaning clearly, which is the purpose of the following argument. The flow of a fluid over a surface, such as the interior walls of a pipe, is opposed by a force which is approximately proportional to the area of the surface, to the density of the fluid and to the square of the velocity at which the fluid is flowing. Therefore, we may write

$$F = kadv^2 \qquad \text{(i)}$$

in which a is the area of the surface in square feet, d is the density of the fluid in pounds per cubic foot, v is the velocity of flow in

feet per second, and F is the opposing force in pounds-weight. The quantity k is sometimes called the *coefficient of friction* of the moving fluid against the walls of the pipe. It depends greatly upon the degree of roughness of the walls.

Consider a pipe of which the length is L feet and of which the inside diameter is D feet. The total area of interior walls of this pipe is πDL square feet, so that, using πDL for a in equation (i), we have $F = k\pi DLdv^2$ for the total opposing force acting on a fluid of density d which flows through the pipe at velocity v. This opposing force is equal to the difference of pressure at the two ends of the pipe multiplied by the sectional area of the bore of the pipe. Therefore, using p for the loss of pressure due to friction, we have

$$F = p\cdot\frac{\pi D^2}{4} = k\pi DLdv^2 \tag{ii}$$

whence

$$p = \frac{4kLdv^2}{D} \tag{71}$$

in which p is the frictional loss of pressure in pounds per square foot in a pipe L feet long and D feet in internal diameter, v is the velocity of the fluid in feet per second, and d is the density of the fluid in pounds per cubic foot. The value of k is about 0.000082 for water in ordinary cast-iron pipes and about 0.0000557 for air in cast-iron pipes.

Elbows and valves cause excessive eddies and therefore an excessive loss of pressure by friction. Methods for estimating the effects of elbows and valves are given in standard works on hydraulics.

Example 1. An iron pipe one foot in diameter and 10,000 feet long discharges 4.25 cubic feet per second of water when the pressure at one end is 6,000 pounds per square foot greater than the pressure at the other end. A discharge of 4.25 cubic feet per second corresponds to a velocity of 5.41 feet per second in the pipe ($= v$). The density of the water is 62½ pounds per

cubic foot ($= d$). Substituting these values in equation (71) and we find for the coefficient k the value 0.000082.

Example 2. Compressed air at a mean pressure of 5.42 atmospheres (density of 0.406 pounds per cubic foot) is forced through 15,000 feet of pipe 8 inches inside diameter at a velocity of 19.32 feet per second with a difference in pressure of 5.29 pounds per square inch between the two ends of the pipe. Reducing these data to the units employed in this chapter and substituting in equation (71), we have for the coefficient k the value 0.0000557.

These two examples indicate the method of determining the approximate value of the coefficient k under given conditions.

134. Definition of the coefficient of viscosity of a fluid. Consider a thin layer of fluid of thickness x between two flat plates AA and BB as shown in Fig. 147 and suppose that the plate AB is moving at velocity v as indicated by the arrows. If the fluid between the plates AB were a viscous liquid like syrup, it is evident that a very considerable force would have to be exerted upon the plate AA to keep it in motion; in fact any fluid whatever, whether liquid or gas, is more or less like syrup in this respect, and the force F with which the motion of the plate is

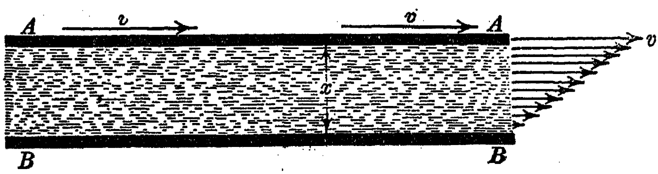


Fig. 147.

opposed by the fluid is proportional to its area a, to its velocity v and inversely proportional to the distance x between the plates. That is

$$F = \eta \cdot \frac{av}{x} \tag{72}$$

in which the proportionality factor η is called the coefficient of viscosity of the fluid.

Examples. The coefficient of viscosity of water at ordinary room temperature is 0.0000215 and the coefficient of viscosity of good machine oil is about 0.00085; F, a, v and x being expressed in terms of the units specified at the beginning of this chapter. It may seem, therefore, that water would be a better lubricant than the oil, but a layer of water would quickly flow out from between a shaft and a bearing surface, whereas a rotating shaft continually carries a fresh supply of a viscous liquid like oil into the space between the shaft and the bearing surface.

135. Flow of a viscous liquid through a small smooth-bore tube. Let R be the radius of the bore of the tube, l the length of the tube, p the difference of pressure of the liquid at the ends of the tube, and v the velocity of the liquid at a point

distant r from the axis of the tube. Consider a cylindrical portion of the moving liquid of radius r and coaxial with the tube. The surface of this cylindrical portion of liquid moves as a *solid rod* through the tube at velocity v. Similarly, the cylindrical surface of radius $r+\Delta r$ moves through the tube as a *hollow shell* at velocity $v-\Delta v$. The layer of liquid between this *rod* and *shell* is under the same conditions of motion as the layer of liquid between the plates AA and BB in Fig. 147. Therefore, writing Δv for v in equation (72), writing Δr for x, and writing $2\pi rl$ for a we have

$$F = \eta \cdot \frac{2\pi rl \cdot \Delta v}{\Delta r}$$

where F is the force which must act on the end of the *rod* to overcome the viscous drag; but this force is equal to the area of the end of the rod multiplied by p, so that

$$\pi r^2 p = \eta \cdot \frac{2\pi rl \cdot \Delta v}{\Delta r}$$

or

$$\frac{dv}{dr} = \frac{p}{2\eta l} \cdot r$$

whence

$$v = \frac{pr^2}{4\eta l} + \text{a constant}$$

but when $r=R$, $v=0$, so that the constant of integration is equal to $-pR^2/4\eta l$ and therefore

$$v = \frac{pr^2}{4\eta l} - \frac{pR^2}{4\eta l} \tag{i}$$

The velocity at each part of the tube is thus determined. To find the volume V of fluid discharged in time τ, consider a section of the tube, Fig. 148. The velocity over all the area, $2\pi r\Delta r$, of the dotted annulus, is v, so that the volume ΔV, flowing across this annulus in time τ, is $\Delta V = 2\pi r\Delta r \cdot v \cdot \tau$. Substituting v from (i), we have

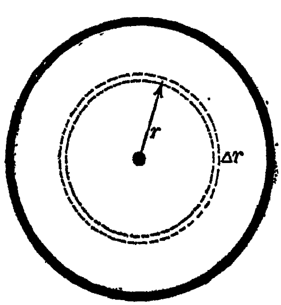


Fig. 148.

$$dV = \frac{\pi p\tau}{2l\eta} r^3 dr - \frac{\pi pR^2\tau}{2l\eta} r dr$$

or

$$V = \frac{\pi p\tau}{2l\eta} \int_0^R r^3 dr - \frac{\pi pR^2\tau}{2l\eta} \int_0^R r dr \tag{ii}$$

Therefore

$$V = \frac{\pi pR^4\tau}{8l\eta}$$

Problems.

164. Find the mean velocity at which water must flow in a canal 20 feet wide and 6 feet deep, in order that the rate of discharge may be 500 cubic feet per second.

How many acres of storage basin would be required to store an amount of water sufficient to maintain this flow of water for 24 hours, the average depth of the water in the storage basin to be 10 feet? Ans. 4.17 feet per second; 99.2 acres.

165. How much work in foot-pounds is required to pump 10,000 cubic inches of water into a reservoir in which the pressure stands at the constant value of 150 pounds per square inch above atmospheric pressure? Ans. 125,000 foot-pounds.

Note. In equation (65) p is to be expressed in pounds per square foot and W' is expressed in foot-pounds of energy per cubic foot of liquid.

166. The velocity of a water jet is 200 feet per second, what is the kinetic energy of the water in foot-pounds per cubic inch Ans. 22.6 foot-pounds per cubic inch.

Note. In equation (67) the density d is expressed in pounds per cubic foot, the velocity v is expressed in feet per second, g is the acceleration of gravity in feet per second per second, and W'' is the kinetic energy in foot-pounds per cubic foot of fluid.

167. Calculate the velocity of efflux of kerosene from a vessel in which the pressure is 52 pounds per square inch above atmosphere pressure. The density of kerosene is 0.03 pound per cubic inch. Ans. 96.2 feet per second.

168. Water flows in a 12-inch main at a velocity of 4 feet per second and encounters a partly closed valve through which the section of the stream is reduced to 0.36 square foot. Calculate the loss of pressure at the valve due to friction. Ans. 0.408 pounds per square inch.

Note. As the water enters the narrow passageway in the valve, its velocity increases by a definite amount, and its pressure falls off accordingly, as explained in Arts. 128 and 129. As the water issues from the narrow passageway, it retains its velocity as a jet flowing through the surrounding water, so that its pressure does not rise again, and the excess of velocity is then destroyed by eddy action. Therefore the loss of pressure through the valve is approximately equal to the drop of pressure due to the increased velocity of the water as it enters the narrow passage.

169. A street water-main 7 inches inside diameter has a throat 3 inches in diameter inserted in it. The flow of water through the pipe is 1½ cubic feet per second and the pressure in the 7-inch pipe is 90 pounds per square inch. What is the pressure in the

throat in pounds per square inch, ignoring friction? Ans. 84 pounds per square inch.

170. The difference in level, h, Fig. 170*p*, is observed to be 6 inches. Calculate the rate of discharge of water through the pipe in cubic feet per second. Ans. 1 cubic foot per second.

Note. The specific gravity of mercury is 13.6, and the tube AB is entirely filled with water above the surface of the mercury.

171. A pair of Pitot tubes is placed in the stream of air issuing from a fan blower, as shown in Fig. 171*p*, and the difference in

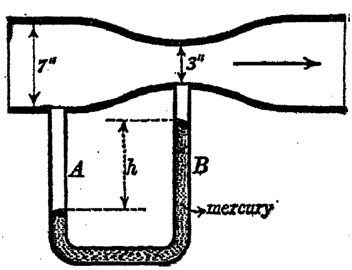


Fig. 170*p*.

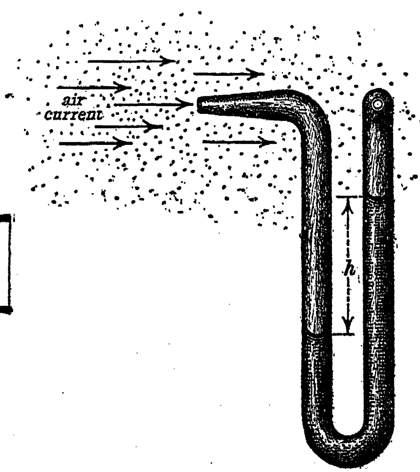


Fig. 171*p*.

level, h, is observed to be $2\frac{1}{2}$ inches, the tubes being filled with water. Calculate the velocity of the air stream. Ans. 72.2 feet per second.

Note. In this problem ignore the compressibility of the air, and assume its specific gravity to be 0.08 pound per cubic foot.

172. A temporary dam of thin boards is built across a stream, a circular hole one foot in diameter is cut through the dam, and the water in the dam rises to a height of 1.5 feet above the center of the hole. Calculate the discharge rate of the stream in cubic feet per second. Ans. 4.67 cubic feet per second.

173. A supply of 10 cubic feet per second of water is to be brought from a reservior to a center of distribution in a small

city. The height of the surface of the reservoir above the point of distribution in the city is 350 feet and it is desired to have an available head of 150 feet at the center of distribution. Required the size of pipe that is necessary to deliver the water, the length of pipe being 16,000 feet. Ans. Diameter of pipe required is 1.336 feet.

174. A water tank is installed for the protection of a factory against fire. The water level in the tank is 75 feet above a certain fire hydrant in the building. The pipe leading from the tank to the fire hydrant consists of 150 feet of 4-inch pipe in which there is one Pratt and Cady check valve, and four short-turn ells; and 200 feet of 3-inch pipe in which there are three short-turn ells. Find the number of gallons of water per second that can be delivered at the hydrant allowing a loss of head of 25 feet in the pipe. Ans. 3.16 gallons per second.

Note. A four-inch Pratt and Cady check valve has a resistance equivalent to 25 feet of four-inch pipe, and short turn ells each have a resistance equivalent to 4 feet of pipe of the same size.

PART II.

THE THEORY OF HEAT.

CHAPTER I.

TEMPERATURE. THERMAL EXPANSION.

1. Thermal equilibrium; temperature. The most important single fact in connection with the study of the phenomena of heat is that a substance, or a system of substances, settles to a quiescent state in which there is no tendency to further change of any kind, when it is left to itself and shielded from all outside disturbing influences. This quiescent state is called a state of *thermal equilibrium.* For example, the various objects in a closed room settle to thermal equilibrium; when a piece of red-hot iron is thrown into a pail of water, the mixture, at first turbulent, becomes more and more quiet and finally reaches a state of thermal equilibrium.

A number of bodies which have settled to a common state of thermal equilibrium are said to have the *same temperature.* Thus, a number of bodies left together in a closed room have the same temperature.

Although the various objects in a closed room are at the same temperature, some of the objects may feel warmer or cooler than the others. A piece of warm metal imparts heat to the hand more rapidly than a piece of wood at the same temperature, and therefore the piece of metal feels warmer than the piece of wood. A piece of cool metal takes heat from the hand more rapidly than a piece of wood at the same temperature, and therefore the piece of metal feels cooler than the piece of wood.

2. Atomic theory of heat and thermodynamics.* In nearly every branch of physical science there are two more or less distinct methods of attack, namely, (*a*) a method of attack in which the effort is made to develop conceptions of the physical processes

*The term thermodynamics is here used in its proper significance, meaning the whole of the theory of heat except those parts which involve the atomic theory.

of nature, and (*b*) a method of attack in which the attempt is made to correlate phenomena on the basis of sensible things, things that can be seen and measured. In the theory of heat, the first method is represented by the application of the *atomic theory* to the study of heat phenomena, and the second method is represented by what is called *thermodynamics.* In the first case one tries to imagine the nature of such a process as the melting of ice or the burning of coal, and in the second case one is content to measure the amount of heat absorbed or given off and to study the physical properties of the substances before and after the change has taken place.

The atomic theory. The theory of heat properly includes the whole of chemistry, and every student of elementary chemistry is familiar with the use of the atomic theory in enabling one to form clear ideas of chemical processes. For example, the burning of hydrogen is thought of as the joining together of atoms of hydrogen and oxygen forming molecules of water vapor. The atomic theory is also of considerable use in giving one clear ideas of the physical properties of substances. Thus a gas is supposed to consist of a great number of particles in violent to-and-fro motion, and the gas exerts pressure against the walls of the containing vessel because of the bombardment of the walls by the rapidly moving molecules of the gas. In addition to these two highly developed branches of the atomic theory (chemistry and the theory of gases), the atomic theory has been applied in a more or less vague but very useful way in the study of a great variety of heat phenomena as exemplified in the following quotation from Tyndall's *Heat A Mode of Motion.** "When a hammer strikes a piece of lead, the motion of the hammer appears to be entirely lost. Indeed, in the early days it was supposed that

*This book appeared about 1875; sixth edition revised in 1880. It should be read by every student who wishes to understand the phenomena of heat in terms of molecular motion. To attempt to develop these general ideas in an elementary text on heat is out of the question; an elementary text on heat must be devoted primarily to thermodynamics. The quotation above is not given in Tyndall's exact words.

what we now call the energy of the hammer was destroyed. But there is no loss. The motion of the massive hammer is transformed into molecular motion in the lead, and here our imagination must help us. In a solid body, although the force of cohesion holds the atoms together, the atoms are supposed, nevertheless, to vibrate within certain limits. The greater the amount of mechanical action invested in the body by percussion, compression, or friction, the greater will be the rapidity and the wider the amplitude of the atomic oscillations.

" The atoms or molecules thus vibrating, and, as it were, seeking wider room, urge each other apart and cause the body of which they are the constituents to increase in volume. By the force of cohesion, then, the molecules are held together; by the force of heat (molecular vibration) they are pushed asunder*; and the relation of these two antagonistic powers determines whether the body is a solid, a liquid or a gas. Beginning with a solid substance, every added amount of heat pushes the molecules more widely apart; but the force of cohesion acts more and more feebly as the distance through which it acts is augmented. Therefore, as the expansive effect of heat grows strong, its opponent, cohesion, grows weak until finally the particles are so far loosened from each other as to be at liberty; not only to vibrate to and fro across a fixed position, but also to roll or glide around each other. Cohesion is not yet entirely destroyed,† but it is modified so as to permit the particles of the substance to glide over each other. *This is the liquid condition of matter.*

" In the interior of a mass of liquid the motion of every molecule is limited and controlled by the molecules which surround it. But when sufficient heat is imparted to a liquid at a point the molecules break the last fetters of cohesion and fly asunder to form a bubble of vapor. At the free surface of a liquid it is

*These two statements by Tyndall are not always true. Thus when ice changes to water contraction takes place and the average distance between the molecules decreases.

†It is a familiar fact that the different parts of a drop of water cling together, or, in other words, the force of cohesion is not entirely absent in water.

easy to conceive that some of the vibrating molecules may escape from the liquid and wander about through space. *Thus freed from the influence of cohesion we have matter in the gaseous form.*"

Thermodynamics. To understand the essential features of the science of thermodynamics, it is necessary to revert to the discussion of work and energy. Whenever a substance, or a system of substances, gives up energy which it has in store, the substance or system of substances always undergoes change. Thus, the fuel which supplies the energy to a steam engine and the food which supplies the energy to a horse undergo a *chemical change;* the steam which carries the energy of the fuel from the boiler to the engine *cools off* or undergoes a *change of temperature* when it gives up its energy to the engine; a clock spring *changes its shape* as it gives up its energy in driving a clock; an elevated store of water *changes its position* as it gives its energy to a water wheel; the heavy fly-wheel of a steam engine does the work of the engine for a few moments after the steam is shut off and the fly-wheel *changes its velocity* as it gives up its energy.

Not only does a substance undergo a change when it gives up energy by doing work, but a substance which receives energy or has work done upon it undergoes a change. Thus, when air is compressed in a bicycle pump work is done on the air and the air *becomes warm*; when work is done upon a coin in rubbing it upon a board, the coin *becomes warm*; when work is done upon a clock-spring in winding it up, the spring *changes its shape*; when work is done in pumping water, the water *changes its position* to an elevated tank or *issues at a high velocity* as from a fire nozzle.

In the chapters on Mechanics, the theory of energy was discussed in connection with mechanical changes only, thermal and chemical changes being carefully ignored. We are now, however, to take up the study of thermal and chemical changes and it is important at the outset to understand two things as follows:

(*a*) Our study is not to be concerned with thermal and chemical actions themselves but with their results. The changes themselves

are, as a rule, extremely complicated. Thus, the details of behavior of the coal and air in a furnace are infinitely complicated. The important practical thing, however, is the amount of steam that can be produced by a pound of coal, and this depends only upon (1) the condition of the water from which the steam is made, that is, whether the water is hot or cold to start with, (2) the condition of the air and of the coal which are to combine in the furnace, (3) the pressure and temperature of the steam which is to be produced, and (4) the condition of the flue gases as they enter the chimney. That is to say, *the only things which it is necessary to consider are the things which relate to quiescent substances.* A quiescent substance may be said to be in a *standing condition* or *state*, and the whole subject of heat (thermodynamics) may be said to refer to *changes of state*, that is, to changes from one quiescent condition to another quiescent condition without regard to the details of action which leads from one quiescent condition to the other. In studying a change of state of a substance, variations of temperature, volume and pressure, changes of chemical composition, and, above all, the energy given to or taken from the substance during the change, are important considerations. Beyond these things but little else is involved in the study of thermodynamics.

(*b*) The other important thing is that in studying thermal and chemical changes we have to do with a new kind of energy. The gravitational energy of an elevated store of water can be wholly converted into mechanical work*, the energy of two electrically charged bodies can be wholly converted into mechanical work (for example by allowing the charged bodies to move towards each other), the kinetic energy of a moving car can be wholly converted into mechanical work, and so on. On the other hand, the energy of the hot steam which enters a steam engine from a boiler *cannot be wholly converted into mechanical work.* Any store

*Any energy which is converted into heat because of friction exists in the form of mechanical energy or work before it is so converted, and this fact must be kept in mind in connection with the statements above given as to the conversion of gravitational and electrical energy into mechanical work.

of energy which can be wholly converted into mechanical work may be called *mechanical energy*. The energy of the hot steam which enters a steam engine from a boiler is called *heat energy*. The important difference between mechanical energy and heat energy, namely, that one can be wholly converted into mechanical work whereas the other cannot, may be clearly understood in terms of the atomic theory: Every particle of a moving car travels in the same direction and all of the particles work together to produce mechanical effect when the car is stopped; the molecules of hot steam, however, fly to and fro in every direction, and no method can be devised whereby the whole of the energy of the molecules of hot steam can be used to produce mechanical effect.

The impossibility of converting the whole of the heat energy of steam into mechanical work by the steam engine does not, of course, refer to the loss of heat by the cooling of the cylinder by the surrounding air. It would seem to be possible to reduce the temperature (and heat energy) of a gas to zero by allowing the gas to expand indefinitely against a piston, but when the pressure of the gas is reduced below the pressure of the surrounding atmosphere work cannot be obtained from the gas by the expansion, on the contrary, work has to be done to produce the expansion. In fact this question of the convertibility of the heat energy of a gas into mechanical work is bound up inextricably with the question as to the temperature and pressure of the surrounding region, as will become evident in Chapters V and VI.

3. The dissipation of energy.* Preliminary statement of the first law of thermodynamics. In the attempt to exclude all thermal changes from the purely mechanical discussion of energy† we were confronted by the fact that friction (with its accompanying thermal changes) is always in evidence everywhere. In every actual case of motion, the moving bodies are subject to

*The *dissipation* of energy is sometimes spoken of as the *degradation* of energy from any form which is wholly available for the doing of mechanical work into heat.

†See *Mechanics*, Art. 55.

friction and to collision, their energy is dissipated, and they come to rest. This dissipation of energy is always accompanied by the generation of heat, and experience shows that the amount of heat generated is equivalent to the energy dissipated (first law of thermodynamics). *A more complete discussion of the first law of thermodynamics is given in Chapter II.* The full significance of the law is that heat is a form of energy, and that the principle of the conservation of energy is applicable not only to mechanical changes but to thermal and chemical changes also. Whenever mechanical energy disappears, an equivalent amount of heat is produced; and whenever heat energy disappears, as in the expansion of the steam against the piston of a steam engine, an equivalent amount of mechanical energy comes into existence.*

It is important to understand that the term "dissipation of energy" refers to the conversion of mechanical energy into heat by friction or collision.† Thus, energy is dissipated in the bearing of a rotating shaft, energy is dissipated when a hammer strikes a nail, and so on. The atomic theory enables one to form a clear idea of the dissipation of energy. Thus the energy of the regular motion of a hammer is converted into energy of irregular‡ molecular motion when the hammer strikes a nail.

*One of the most important steps in the establishment of the principle of the conservation of energy in its general form, which includes heat energy, was made by Count Rumford in his experiments on the generation of heat in the operation of boring cannon. The results of these experiments were published in the *Philosophical Transactions* for 1799. The first clear statement of the principle of the conservation of energy in its general form was published in 1842 by Julius Robert Mayer. The celebrated experiments of Joule on the heating of water by the dissipation of work were commenced in 1840. These experiments are described on pages 274–278 of Edser's *Heat for Advanced Students*. The most accurate investigation on the heating of water by the dissipation of mechanical energy up to the present time is the work of Rowland in 1879. Rowland's experiments, which are discussed in Chapter II, are described in detail on pages 278–281 of Edser's *Heat for Advanced Students*, published by Macmillan & Co., London, 1908.

†Mechanical energy is also dissipated in a wire in which an electric current is flowing.

‡A substance in thermal equilibrium exhibits no visible motion and therefore a state of thermal equilibrium has been called a quiescent state. Very violent molecular motion is supposed, however, to exist when a substance is in thermal

4. Imparting of heat to a substance and the observable effects produced thereby. Heat may be imparted to a substance by the dissipation of mechanical energy in the substance. Thus, heat may be imparted to a coin by rubbing it on a board. Heat may also be imparted to a substance by placing it in contact with a hotter substance. Thus, heat is imparted to a tea-kettle which is placed upon a hot stove. When heat is imparted to a substance, the following observable effects may be produced:

(1) The temperature of the substance may rise. Thus, when heat is imparted to a piece of iron or to a vessel of cold water, the temperature of the iron or water rises. When, however, heat is imparted to ice at its melting point, some of the ice is converted into water but no change of temperature is produced.

(2) The substance may expand or contract. Most substances expand with rise of temperature. There are, however, several exceptions to this general rule. Thus, water contracts as its temperature rises from the freezing point to about 4°C. at which temperature the density of water is a maximum. The expansion of a gas with rise of temperature is very much greater than the expansion of a liquid or a solid.

equilibrium but this molecular motion is of the same average character in every part of the substance.

It requires some power of imagination to think of a substance as being composed of a great number of small particles (molecules) in incessant and irregular motion, and to think of the energy of a moving hammer as still existing by virtue of an increased violence of molecular motion after a hammer blow. Every student of physics should see the irregular and incessant to-and-fro motion of very fine particles suspended in water, using a good microscope. This motion was discovered by the English botanist, Brown, in 1827, and it is called the *Brownian motion*. The Brownian motion is the irregular molecular motion of the water rendered visible (and greatly reduced in amplitude) by the small suspended particles.

To see the Brownian motion, grind a small amount of insoluble carmine in a few drops of water by rubbing with the finger in a shallow dish, place a drop of the mixture on a microscope slide, and use a magnifying power of about 400 diameters. The particles in India ink are much finer than the particles of carmine and a higher magnifying power is required to see them.

A very interesting discussion of the present position of the atomic theory is given by Ernest W. Rutherford in his address before the Physics Section of the Winnepeg meeting of the British Association for the Advancement of Science. This address is published in *Science*, new series, Vol. 30, pages 289–302, September 3, 1909.

(3) The substance may melt or vaporize without change of temperature. Thus, when heat is imparted to ice at the melting point, part of the ice is converted into water without increasing in temperature. When the water in a tea-kettle begins to boil, the continued imparting of heat to the tea-kettle from the stove converts a portion of the water into steam without increasing its temperature.

(4) The substance may be dissociated. For example, wood is converted into charcoal and a smoky gas when it is heated. Lime-stone (calcium carbonate) is broken up into quicklime (calcium oxide) and carbon dioxide gas when it is heated in a lime kiln.

(5) The subtance, if sufficiently heated, gives off light.

(6) The substance may exhibit certain electrical phenomena.*

5. Thermal expansion of gases. Gay Lussac's Law. When a number of closed vessels containing different gases all at the same pressure are carried from a cool cellar, for example, to a warm room, they all suffer the same rise of temperature, *and all of the gases show the same increase of pressure.* That is to say, all gases follow the same law of increase of pressure with increase of temperature, the volumes of the containing vessels being constant. This fact was discovered by Gay Lussac and it is called *Gay Lussac's Law.* Following are two precise statements of Gay Lussac's Law:

(*a*) When equal volumes of various gases are heated under constant pressure, they all suffer the same expansion for the same rise in temperature.

(*b*) When various gases under the same initial pressure are heated and not allowed to expand, they all suffer the same increase of pressure for the same rise in temperature.

Note. When first discovered, Gay Lussac's Law was thought to be exactly true. Very careful measurements, however, show perceptible differences of expansion of various gases.

*See Franklin and MacNutt's *Elements of Electricity and Magnetism*, appendix C.

6. The measurement of temperature. To measure a thing is to divide it into equal (congruent) parts and to count the parts.* One cannot of course divide a force into congruent parts, and therefore in a certain fundamental sense forces cannot be measured. Neither can one divide a temperature into congruent parts, and therefore in a certain fundamental sense temperatures cannot be measured. Indeed such things as temperature and force can be measured only in terms of their effects. One might, for example, take a portion of any gas at the temperature of melting ice, and measure the pressure of the gas (without change of volume) at the temperature of boiling water, at the temperature of melting lead, etc., and any one of these temperatures could then be specified numerically by giving the pressure of the gas at that temperature, or, better, by giving the ratio of the pressure of the gas at that temperature to the pressure of the gas at the temperature of melting ice, the volume of the gas being unchanged. According to this scheme the ratio of two temperatures is the ratio of the pressures of a constant volume of gas at the respective temperatures. Thus, if p and p' are the pressures of a constant volume of a gas at temperatures T and T' respectively, then we have by definition

$$\frac{T}{T'} = \frac{p}{p'} \tag{1}$$

This provisional definition of temperature ratios will be found later to coincide with the thermodynamic definition of temperature ratios.

The air thermometer.† The air thermometer is a device for measuring the ratio of two temperatures by observing the pressures of a constant volume of dry air at the respective temperatures. The essential parts of the air thermometer are shown in Fig. 1. The glass or porcelain bulb A contains dry air, and the pressure of the air is measured by a syphon barometer BB (or

*See page 12.

†The hydrogen thermometer is the accepted standard.

open tube manometer). The short arm of the barometer at a communicates with the bulb A through a tube of fine bore. A movable reservoir R containing mercury communicates with the barometer through a flexible rubber tube and serves to bring the surface of the mercury at a to a marked point near the end of the fine bore tube, thus keeping the volume of the enclosed air sensibly constant.

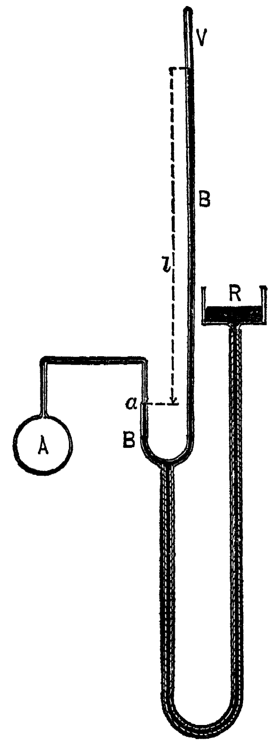


Fig. 1.

The bulb A is brought to temperature T and the height of the mercury column l is measured; the bulb A is then brought to temperature T' and the height l' of the mercury column is again measured. Then, according to equation (1), we have

$$\frac{T}{T'} = \frac{l}{l'}$$

Standard temperatures. Experiment shows that the temperature of pure melting ice and the minimum temperature of pure steam at a given pressure are invariable. These temperatures, at standard atmospheric pressure of 760 millimeters of mercury, are taken as the standard temperatures in thermometry. They are called the ice point (I) and the steam point (S) respectively. By placing the bulb of an air thermometer in melting ice and then in a steam bath (steam at normal atmospheric pressure), the pressures of the enclosed air are found to be in the ratio of 1 : 1.367. Therefore, according to the above provisional definition of temperature ratios, we have

$$\frac{S}{I} = 1.367 \tag{2}$$

If the centigrade scale (see Art. 7) is used, the arbitrary value

100 is assigned to the difference $S - I$, so that

$$S - I = 100 \tag{3}$$

From these two equations we find that $S = 373$ and $I = 273$ approximately. The values of S and I are thus known, and any other temperature may be determined by measuring its ratio to I (or to S) by means of the air thermometer. Temperatures measured in this way are called *absolute temperatures*.

Formulation of Gay Lussac's Law and Boyle's Law. When temperature is measured by the air thermometer, then the pressure of a constant volume of *any* gas is proportional to the absolute temperature of the gas, or the volume of a gas is proportional to the absolute temperature, if the gas is allowed to expand so as to keep its pressure constant as the temperature rises.

According to Boyle's law,* the pressure of a gas is inversely proportional to its volume if the temperature is constant, and according to Gay Lussac's law, the pressure is directly proportional to the absolute temperature if the volume is constant. It follows from this that the product of pressure and volume is proportional to the absolute temperature, that is, we may write

$$pv = R'T \tag{4a}$$

in which p is the pressure of a gas, v is the volume of the gas, T is the absolute temperature of the gas, and R' is a constant which depends upon the amount of gas and upon the units in terms of which pressure and volume are expressed. A more satisfactory form of the above equation is

$$pv = MRT \tag{4b}$$

in which M is the mass of the gas in grams and R is a constant which is independent of the amount of the gas.

7. The mercury-in-glass thermometer.† The most convenient device for measuring temperature is the ordinary mercury-in-

*See *Mechanics*, Art. 101.

†A good description of the construction of a mercury-in-glass thermometer is given on pages 4–14, and special forms of thermometer for indicating maximum

glass thermometer with which every one is familiar. A glass tube AB, Fig. 2, of fine uniform bore, with a bulb at one end, is filled with mercury at a temperature somewhat above the steam point and the tube is sealed at A. As the instrument cools, the mercury contracts more rapidly than the glass and thus only partly fills the stem. The instrument is then placed in an ice bath and the position of the surface of the mercury in the stem is marked at I. Then the instrument is placed in a steam bath at standard atmospheric pressure, and the steam point is marked at S.

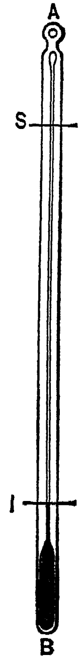


Fig. 2.

In the centigrade scale (*Celsius*),* which is the scale universally used in scientific work, the distance SI is divided into 100 equal parts, which divisions are continued above S and below I. These marks are numbered upwards beginning at I which is number zero. The marks below I are numbered negatively from I.

Any temperature is specified by giving the number of the mark at which the mercury stands when the thermometer is brought to that temperature. For example, 65° C. (read *sixty-five degrees Centigrade*) is the temperature at which the mercury in a mercury-in-glass thermometer stands at mark number 65 of the centigrade scale.

Mercury-in-glass temperatures. The indications of an accurately constructed mercury-in-glass thermometer are slightly different from air thermometer temperatures

temperatures and minimum temperatures are described on pages 18–20 of Edser's *Heat for Advanced Students*, Macmillan & Company, London, 1908.

A device for measuring very high temperatures is called a pyrometer. A good discussion of the older methods for measuring high temperatures is given in Burgess's translation of *High Temperature Measurements* by Le Chatlier and Boudouard, John Wiley & Sons, New York, 1904. An excellent discussion of optical methods for measuring high temperatures is given by Waidner and Burgess, *Bulletin of the Bureau of Standards*, Vol. I, pages 189–254, February, 1905.

*The only other thermometer scale of which mention need be made is that of Fahrenheit in which the distance SI is divided into 180 equal parts, which divisions are continued above S and below I. These marks are numbered upwards beginning with the thirty-second mark below I which is number zero. The marks below zero are numbered negatively.

(reckoned from ice point) because of the irregularities in the expansion of mercury and glass, and temperature values as indicated by an accurate mercury-in-glass thermometer made of a standard variety of glass are called mercury-in-glass temperatures.

The following table shows air-thermometer temperatures (reckoned from ice point) and mercury-in-glass temperatures (Jena normal glass) corresponding to hydrogen-thermometer temperatures reckoned from the ice point. All three thermometers agree, of course, at ice point and at steam point, and the differences for the intervening temperatures depend upon irregularities of expansion. Thus, the difference between the hydrogen thermometer temperatures and the air thermometer temperatures show that these gases do not both expand in exactly the same way with rise of temperature, and another difference between the hydrogen and the air thermometers which does not appear in the table is that the ratio of steam temperature to ice temperature as measured by the hydrogen thermometer is slightly different from the ratio as measured by the air thermometer.

TABLE.*

COMPARISON OF HYDROGEN, AIR AND MERCURY-IN-GLASS TEMPERATURES.

Hydrogen thermometer temperatures (reckoned from ice point)	Air thermometer temperatures (reckoned from ice point)	Mercury-in-glass temperatures (Jena Normal Glass)
0.°	0.°	0.°
10.	10.007	10.056
20.	20.008	20.091
30.	30.006	30.109
40.	40.001	40.111
50.	49.996	50.103
60.	59.990	60.086
70.	69.986	70.064
80.	79.987	80.041
90.	89.990	90.018
100.	100.	100.

Standard thermometers. It is of course impossible to construct a thermometer so that the bore of the stem is perfectly uniform, and slight errors are always made in the location of the ice and steam points and in the marking of the divisions on the stem.

*From Landolt and Börnstein's *Physikalisch-Chemische Tabellen*, page 93.

A standard thermometer is a thermometer of which the errors have been determined* so that the true mercury-in-glass temperature corresponding to any given reading is known. No thermometer which has not been standardized is to be depended upon for work of even moderate accuracy.†

8. Thermal expansion of liquids and solids. In general, liquids and solids expand with rise of temperature. This is illustrated by the fact that a long line of steam pipe has to be provided with a telescope joint to allow for expansion and contraction inasmuch as the temperature of the pipe is apt to be changed at any time from ordinary air temperature to steam temperature when the steam is turned on, or from steam temperature to air temperature when the steam is turned off. The movement of the mercury column in the stem of a thermometer shows that mercury expands more rapidly than glass as the temperature rises. The expansion of the glass causes the bulb to grow larger but the greater expansion of the mercury causes the mercury to rise in the stem. The expansion of a gas (at constant pressure) is very much greater than the expansion of a liquid or solid, and all gases expand very nearly alike (Gay Lussac's Law), whereas every liquid and every solid exhibits characteristic peculiarities, expanding more rapidly at certain temperatures than at others, and in some cases actually contracting with rise of temperature. Most liquids exhibit marked irregularities of expansion near their freezing points. Thus, water contracts as it is heated from 0°C. to 4°C. at which temperature the volume of a given mass of water is a minimum or its density is a maximum; and beyond 4°C. water increases in volume with rise of temperature, at first slowly and then more and more rapidly as the temperature rises. The ordinates of the curve *W* in Fig. 3

*A good discussion of the standardization of a mercury-in-glass thermometer is given on pages 23–38 of Edser's *Heat for Advanced Students*. A discussion of the use of a mercury-in-glass thermometer is given on pages 140–143 of Franklin, Crawford and MacNutt *Practical Physics*, Vol. I.

†A well-made thermometer can be sent to the United States Bureau of Standards Washington, D. C., where it will be standardized for a small fee.

show the volumes at various temperatures of an amount of water whose volume at 0°C. is equal to unity, and the ordinates of the curve *M* show the volumes at various temperatures of an amount

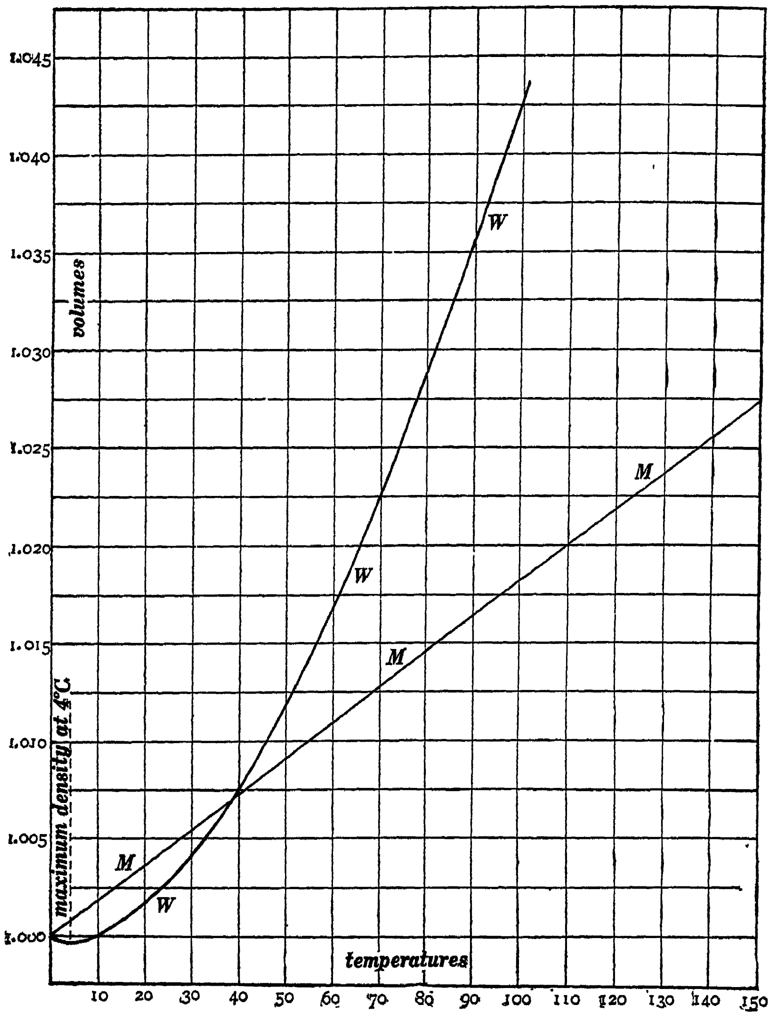


Fig. 3.

of mercury whose volume at 0°C. is equal to unity. The curve *M* is not a straight line, but its curvature is imperceptible in so small a figure.

*Coefficient of linear expansion.** The ordinates of the curve *cc* in Fig. 4 represent the lengths of a metal bar at various temperatures, L_o being the length at o°C. and L_t being the length at t°C. The curvature of *cc* is greatly exaggerated in this

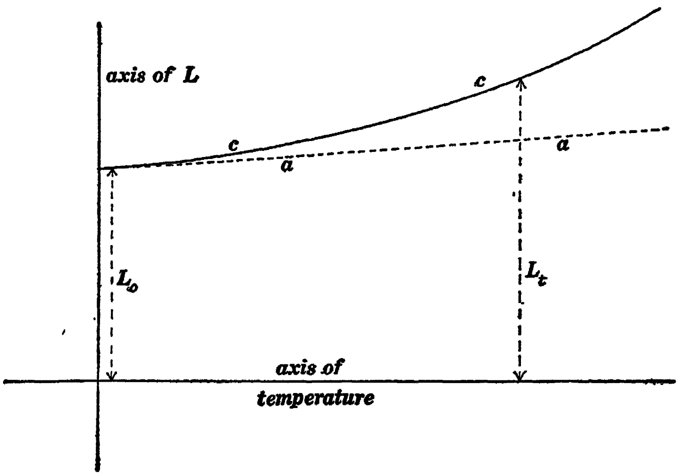


Fig. 4.

figure, and for most practical purposes the portion *cc* of the curve may be treated as a straight line, that is to say, the increase of length $L_t - L_o$ of the bar from o°C. to t°C. may be considered to be proportional to t. This increase in length is also proportional to the initial length L_o of the bar because each unit length of bar expands by the same amount. Therefore we may write

$$L_t - L_o = \alpha L_o t$$

or

$$L_t = L_o(1 + \alpha t) \tag{5}$$

The proportionality factor α is called the *coefficient of linear expansion of the substance*, and it is equal to the increase of length of

*Coefficients of linear expansion and coefficients of cubic expansion of a great variety of substances are given in John Castell-Evans' *Physico-Chemical Tables*, and in Landolt and Börnstein's *Physikalisch-Chemische Tabellen*. Every student of physics and chemistry and every engineer should have access to these tables.

a bar of which the initial length is unity when the temperature of the bar is raised one degree.

In calculating the length of a bar from equation (5), greater accuracy may be obtained by treating the curve of expansion as if it were the straight line *aa* in Fig. 5 instead of the straight line

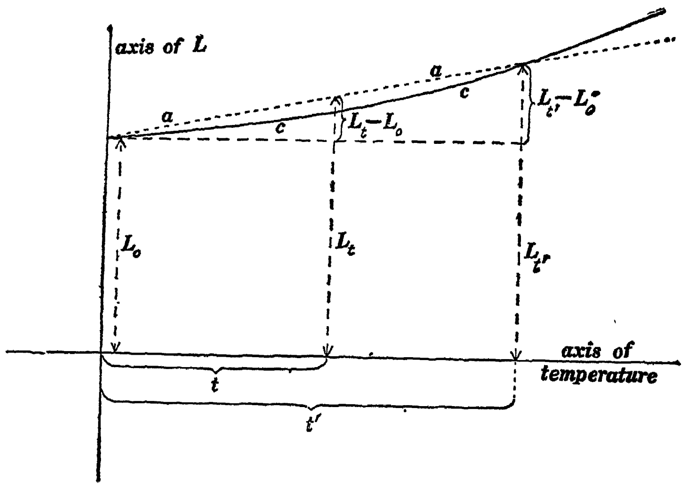


Fig. 5.

aa in Fig. 4. To do this, the value of α is calculated from equation (5) in terms of the observed lengths of the bar at zero and at any chosen temperature t', giving

$$\alpha = \frac{L_{t'} - L_o}{L_o t'} \tag{6}$$

When so determined, the value of α is called the *mean coefficient of expansion for the range of temperature from zero to* t', and when this value of α is used in equation (5) the length of the bar at any temperature is represented by the ordinate of the curve *aa* in Fig. 5.

Some idea of the accuracy with which equation (5) represents the expansion of a substance may be obtained from the following example. The mean coefficient of linear expansion of a certain sample of annealed steel between 0°C. and 100°C. was found by Benoit to be 1.0877×10^{-5} per degree centigrade, whereas the coefficient of linear expansion of the steel at 0°C. [value of α in equation (5) to give the

dotted line in Fig. 4] was found to be 1.0354×10^{-5}. For purposes of very accurate calculations, the expansion of this sample of steel may be represented by the formula

$$L_t = L_o(1 + \alpha t + \alpha' t^2) \tag{7}$$

in which α is equal to 1.0354×10^{-5} and α' is equal to 5.23×10^{-9}. To use this equation is to consider the curve of expansion to be a parabola. It must be remembered however, that the use of equation (7) gives approximate results, even when it is applied to the identical sample of steel which was used in the determination of α and α', especially if the calculation is made for a very high temperature; and of course, calculations based upon equations (5) are only approximate.

Coefficient of cubic expansion. Let V_0 be the volume of a given substance at 0°C. and V_t its volume at t°C. The increase of volume from 0°C. to t°C. is $V_t - V_0$, and it is accurately proportional to the initial volume of the substance and approximately proportional to the rise of temperature t; that is

$$V_t - V_0 = \beta V_0 t$$

or

$$V_t = V_0(1 + \beta t) \tag{8}$$

The proportionality factor β is called the *coefficient of cubic expansion of the substance.* It is equal to the increase of volume of a portion of the substance of which the initial volume is unity when the temperature of the substance is raised one degree.

The value of β is always determined from equation (8) using the observed volume, V_0, at zero, and the observed volume $V_{t'}$ at t'°C., giving

$$\beta = \frac{V_{t'} - V_0}{V_0 t'}$$

The value of β so determined is the *mean coefficient of cubic expansion of the substance for the range of temperature from* 0°C. to t'°C.

Relation between coefficients of linear and cubic expansion. The coefficient of cubic expansion of a substance is equal to three times its coefficient of linear expansion. Consider a cube of the substance of which the length of edge at 0°C. is L_0. At t°C. the length of edge is $L_0(1 + \alpha t)$. The volume at 0°C. is $V_0 = L_0^3$, and the volume at t°C. is $V_t = L_0^3(1 + \alpha t)^3$, or $V_t = L_0^3(1 + 3\alpha t$

$+ 3\alpha^2t^2 + \alpha^3t^3)$. The terms $3\alpha^2t^2$ and α^3t^3 are negligible, and therefore writing V_o for L_o^3, we have

$$V_t = V_o(1 + 3\alpha)$$

Comparing this equation with equation (8), it is evident that β is equal to 3α.

Peculiarities of expansion of solids. Solids show irregularities of expansion which are in some cases as marked as the irregularities of expansion of liquids near their freezing points. These irregularities occur at what are called *transition temperatures*, a transition temperature for a given substance being a temperature below which the substance is in one crystalline form and above which the substance is in another crystalline form. The most familiar example of a transition temperature is the so-called temperature of recalescence of steel.*

Solids exhibit other peculiarities of expansion which are not exhibited by liquid and gases. Thus, many solid substances do not expand promptly with rise of temperature or contract promptly with fall of temperature, the ultimate change of dimensions corresponding to a given change in temperature requiring in some cases days or even months before it is established. The best known example of this time-lag of expansion is furnished by ordinary glass. The mercury column of a mercury-in-glass thermometer which has been kept for a long time at room temperature and which is suddenly brought to steam temperature rises at first too high, and as the bulb slowly expands to the ultimate size which corresponds to steam temperature the mercury column slowly drops to its correct position.

A most interesting substance is the non-expansible nickel-steel alloy which was discovered by Guillaume, a nickel-steel containing 36% of nickel, and known as *invar*. Its coefficient of expansion is less than one tenth of that of ordinary steel. The increase in length of a meter scale made of invar when it is heated from 0°C. to 100°C. would be a little less than 0.1 of a millimeter,

*See Art. 28.

whereas a meter scale made of ordinary steel would increase in length by about 1.3 millimeters for the same rise of temperature. This alloy, invar, is very sluggish in its expansion and contraction. When the increase of temperature is small the increase of length does not reach its full value for the space of two months. Therefore when a bar of invar is subjected to fluctuations of temperature which are neither very large nor very long continued the change of length of the bar is extremely small and for many purposes negligible.

9. Regnault's method for determining the expansion of water and of mercury. The density of any given substance is usually determined by weighing equal volumes of that substance and of water, thus finding the specific gravity of the substance (number of times greater its density is than the density of water at the same temperature) so that if the density of the water is known at the given temperature, the density of the substance may be found. A measuring vessel or graduate, such as is used by chemists, is usually standardized by weighing the water or mercury required to fill it, whence, if the density of the water or mercury is known, the volume of the vessel can be calculated. Carefully determined values of the density of water at various temperatures and of the density of mercury at various temperatures are, therefore, of great importance. The densities of water and mercury have been accurately determined at a given temperature by weighing measured volumes of water and mercury; and the densities at other temperatures have been determined by a method due originally to Regnault. In order to understand Regnault's method, it is necessary to establish the relationship between the volume of a substance at different temperatures and its density at different temperatures. Consider a substance of mass m of which the volume at 0°C. is V_0 and the volume at t°C. is V_t. The density of the substance at 0°C. is

$$d_0 = \frac{m}{V_0} \tag{i}$$

and the density of the substance at t°C. is

$$d_t = \frac{m}{V_t} \qquad \text{(ii)}$$

Dividing equation (i) by equation (ii), member by member, we have

$$\frac{d_o}{d_t} = \frac{V_t}{V_o} \qquad \text{(iii)}$$

from which the volume V_t can be calculated when V_0 is known and when the ratio d_0/d_t has been determined.

Regnault's method for determining the ratio d_0/d_t, is as follows:*
Two tubes, A and B, open at top and connected by an air tube C at bottom, are filled with the liquid as shown in Fig. 6. The tube A is placed in a bath at temperature t°C. and the tube B is placed in a bath at temperature 0°C., and the vertical distances l_0 and l_t are measured. Then

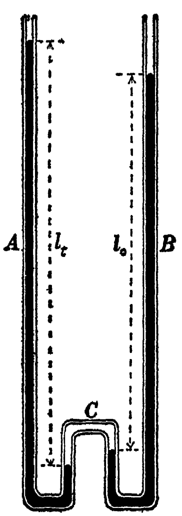


Fig. 6.

$$\frac{d_o}{d_t} = \frac{l_t}{l_o} \qquad \text{(iv)}$$

This equation is eivdent when we consider that the pressure of the air in C exceeds the outside air pressure by the amount $l_0 d_0 g$ or by the amount $l_t d_t g$, where g is the acceleration of gravity, as explained in the Chapter on Hydrostatics. Therefore

$$l_o d_o g = l_t d_t g$$

from which equation (iv) follows at once. The following tables give the values of density (grams per cubic centimeter) and specific volume (cubic centimeters per

*A full discussion of this method is given in Edser's *Heat for Advanced Students*, pages 71–80. The discussion here given is merely an outline.

gram) of pure air-free water and of pure mercury at various temperatures as determined by Thiessen, Scheel and Marek.*

DENSITIES AND SPECIFIC VOLUMES OF WATER.

Temperature.	Grams per cubic centimeter.	Cubic centimeters per gram.	Temperature	Grams per cubic centimeter.	Cubic centimeters per gram.
0°C.	0.999874	1.000127	55	0.98579	1.01442
5	0.999992	1.000008	60	0.98331	1.01697
10	0.999736	1.000265	65	.098067	1.01871
15	0.999143	1.000857	70	0.97790	1.02260
20	0.998252	1.001751	75	0.97495	1.02569
25	0.997098	1.002911	80	0.97191	1.02890
30	0.995705	1.004314	85	0.96876	1.03224
35	0.994098	1.005936	90	0.96550	1.03574
40	0.99233	1.00773	95	0.96212	1.03938
45	0.99035	1.00974	100	0.95863	1.04315
50	0.98813	1.01201			

DENSITIES AND SPECIFIC VOLUMES OF MERCURY.

Temperature.	Grams per cubic centimeter.	Cubic centimeters per gram.	Temperature.	Grams per cubic centimeter.	Cubic centimeters per gram.
0°C.	13.5956	0.0735532	110	13.3284	0.0750276
10	13.5709	0.0736869	120	13.3045	0.0751624
20	13.5463	0.0738207	130	13.2807	0.0752974
30	13.5218	0.0739544	140	13.2569	0.0754325
40	13.4974	0.0740882	150	13.2331	0.0755679
50	13.4731	0.0742221	160	13.2094	0.0757035
60	13.4488	0.0743561	170	13.1858	0.0758394
70	13.4246	0.0744901	180	13.1621	0.0759755
80	13.4005	0.0746243	190	13.1385	0.0761120
90	13.3764	0.0747586	200	13.1150	0.0762486
100	13.3524	0.0748931	210	13.0915	0.0763857

10. Some phenomena dependent upon thermal expansion. The winds of the earth constitute the most important group of phenomena dependent upon thermal expansion. The radiation from the sun penetrates through the upper portions of the atmosphere and reaches the ground where it heats the lower layers of the air causing them to expand. After a considerable amount of this warm air has accumulated near the ground, it gets started upwards at a given point, and a chimney-like effect is developed which draws the surrounding warm air into the base of the rising

*These tables are taken from the more complete tables given in Landolt and Börnstein's *Physikalisch-Chemische Tabellen.*

column. In the region near the base of the rising column of warm air the pressure of the air is low because of the low density of the great volume of overlying warm air. The region near the base of the rising air column is therefore called a region of "low barometer." The wind blows towards such a region of "low barometer" from all sides, constituting what is called a cyclone.* The movement of the air which is here described is intensified by the formation of water vapor near the ground, because water vapor is only about half as heavy, volume for volume, as dry air.

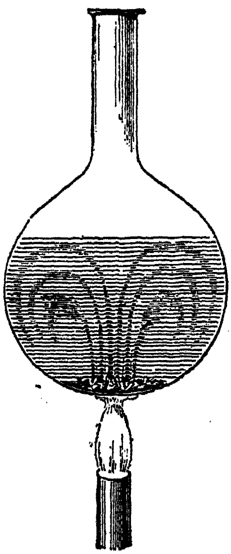


Fig. 7.

The draught of a chimney is due to the fact that the hot gases in the chimney are lighter than the surrounding atmosphere. It is a familiar experience that a chimney may not draw at all after it has stood idle during the summer months, the temperature of the air in the chimney may then be the same or perhaps even less than the temperature of the surrounding air. As soon as the chimney becomes warm, it produces a draught.

The circulation of a liquid due to local heating may be shown in a striking way by heating a flask of cold water, using a very small Bunsen flame, as shown in Fig. 7; a few crystals of magenta being placed at the bottom of the flask. The water at the bottom dissolves the cyrstals of magenta and becomes colored. This colored water expands due to the heat of the Bunsen flame and rises, causing a circulation of the water in the flask which is indicated by the streamers of colored liquid.

*See Art. 123 of the chapter on Hydraulics. The word *cyclone* is used popularly for the extremely violent local storms which are properly called *tornadoes*. The cyclone covers many thousands of square miles of country and the air movements are widespread and usually of moderate intensity.

The circulation of a liquid or gas as above described causes the rapid distribution of heat throughout the liquid or gas; heat is conveyed from one point to another by the moving fluid, and the process is called *convection.*

Problems.

1. Suppose that the average velocity of the molecules of a gas at a given temperature is 40,000 centimeters per second. Find the increase of the average velocity when 10 joules of mechanical energy are dissipated in one gram of the substance; assume that all of the heat energy in the substance is kinetic energy of molecular motion. Ans. 2426 centimeters per second.

Note. The average molecular velocity of a gas is of course zero inasmuch as velocities occur equally in every direction. What is here referred to is the square-root of-the-average-square of the molecular velocity, that is to say, a velocity whose square multiplied by the mass of the substance and divided by 2 gives the total kinetic energy of molecular motion in ergs, velocity being expressed in centimeters per second and mass in grams.

2. The air in the bulb of an air thermometer has a pressure of 750 millimeters when the bulb is placed in a steam bath (at standard atmospheric pressure) and a pressure of 1203 millimeters when the bulb is placed in a bath of lead at its melting point. What is the temperature of melting lead reckoned from ice point? Ans. 325°C.

Note. In solving this problem ignore the slight increase of volume of the air thermometer bulb between steam temperature and the temperature of melting lead. In the accurate use of the air thermometer the expansion of the bulb must be taken account of.

3. The pressure of the air in an air thermometer bulb at the steam point (at standard atmospheric pressure) is 1.367 times as great as the pressure of the air in the bulb when it is in an ice bath. The difference between ice temperature and steam temperature is given the arbitrary value of 180° on the Fahrenheit scale. Find the absolute temperature of the freezing point in Fahrenheit degrees. Ans. 491°F.

4. (*a*) Reduce to Fahrenheit the following centigrade temperatures: 45°, 12°, and −20°. (*b*) Reduce to centigrade the

following Fahrenheit temperatures: 212°, 72°, 32° and −30°. Ans. 113°F.; 53.6°F.; −4°F.; 100°C.; 22.2°C.; 0°C.; −34.4°C.

5. At what temperature do Fahrenheit and Centigrade thermometers give the same reading? Ans. −40°.

6. The stem of a thermometer has upon it a scale of equal parts, and the ice point and the steam point of the thermometer are observed to be at a distance of 92.6 of these divisions apart. (*a*) At what point will the mercury stand at a temperature of 67°C.? (*b*) At what point will the mercury stand at a temperature of 120°F.? Ans. (*a*) 62.04. (*b*) 45.3.

7. Suppose a thermometer stem to be divided into 140 equal spaces between the ice point and the steam point and suppose the marks to be numbered upwards from the tenth division below the ice point, making the ice point No. 10. Reduce the following readings of this thermometer to centigrade and to Fahrenheit: 150°, 70°, 0°, and −20°. Ans. 107°.1C.; 42°.8C.; −7°.1C.; −21°.4C.; 224°.9F.; 109°.1F.; 19°.1F.; −6°.6F.

8. An amount of gas at 15°C. has a volume of 120 c.c. Find its volume at 87°C., the pressure being unchanged. Ans. 150 cubic centimeters.

9. A volume of hydrogen at 11°C. measures 4 liters. The gas is heated until its volume is increased to 5 liters without changing the pressure. Find the new temperature. Ans. 82°C.

10. A flask containing air at 760 millimeters pressure is corked at 20°C. Find the pressure of the air in the flask after it has stood in a steam bath at 98°C., neglecting the slight increase of volume of the flask. Ans. 962 millimeters.

11. The density of dry air at 0°C. and 760 millmeters is 0.001293 gram per cubic centimeter. What is the volume of 25 grams of air at 25°C. and at a pressure of 730 millimeters? Ans. 21,970 cubic centimeters.

Note. When the pressure, volume and temperature of a gas all change, calculations can be most easily made by using the equation

$$\frac{pv}{T} = \frac{p'v'}{T'}.$$

This form of equation obviates any consideration of the values of the constants M and R in equation (4*b*).

12. A quantity of gas is collected over mercury in a eudiometer tube. The volume of the gas is observed to be 50 cubic centimeters, its temperature is 10°C., the level of the mercury in the tube is 10 centimeters above the level of the mercury in the basin, and a barometer shows that the atmospheric pressure is 750 millimeters. Find the volume the gas would occupy at 0°C. and 760 millimeters. Ans. 41.25 cubic centimeters.

13. The material of a toy balloon weighs 50 grams, and the gas in the balloon, consisting partly of carbon dioxide, would have at 0°C. and 760 millimeters pressure, a density of 0.00132 gram per cubic centimeter. The volume of the gas in the balloon is one cubic meter. Find the temperature of the enclosed gas which will barely suffice to buoy up the balloon, the outside air being at 0°C. and 760 millimeters pressure. Ans. 17°C.

14. The air in a tall building has an average temperature of 20°C. and the outside air has a temperature of −10°C. The average density of the air inside of the building is that of dry air at 20°C. and 760 millimeters pressure, and the average density of the air outside is that of dry air at −10°C. and 760 millimeters pressure. The inside and outside pressures are equal at a point which is 80 meters above the ground floor. Find their difference at the ground floor. Ans. approximately 0.805 millimeters. (The correct result is 0.822 millimeters.)

Note. The density of the air in the building would not be uniform even if the temperature were everywhere the same because the pressure cannot be the same from top to bottom. The same is true of the outside air, its density would not be uniform even though its temperature were everywhere −10°C. The problem is to be solved approximately on the assumption that the density of the outside air is uniform and that the density of the inside air is uniform.

The law of decrease of pressure with increase of altitude in the atmosphere when the density is not assumed to be uniform is derived as follows for the case in which the temperature is uniform; let p be the pressure of the air at a chosen point (the origin of coordinates) and δ its density. Then the pressure at a point distant Δx below the origin is

$$p+\Delta p=p+\delta g\cdot\Delta x \qquad \text{(i)}$$

so that

$$\Delta p=\delta g\cdot\Delta x \qquad \text{(ii)}$$

The density of the air (temperature assumed to be constant) is directly proportional to the pressure according to Boyle's Law. Therefore we may write

$$\delta = kp \tag{iii}$$

in which k is a proportionality factor. Substituting this value of δ in equation (ii), we have

$$\Delta p = kpg \,.\, \Delta x \tag{iv}$$

whence, using differential notation, we have

$$\frac{dp}{p} = kg \cdot dx \tag{v}$$

whence, by integration, we find

$$\log p = kgx + \text{a constant} \tag{vi}$$

or

$$p = Ce^{kgx} \tag{vii}$$

that is to say, the pressure in an atmosphere of uniform temperature increases according to the exponential function Ce^{kgx} of the distance x below the origin of coordinates, where C is the value of the pressure at the origin, k is a constant which appears in the expression of Boyle's Law [see equation (iii)], and g is the acceleration of gravity.

15. The pressure inside of a chimney at its base is less than the outside atmospheric pressure by an amount equivalent to a column of water ½ inch in height. The chimney is 125 feet high. Find the average temperature of the gases in the chimney under the following assumptions: The gas in the chimney would have a density of 0.00132 gram per cubic centimeter at 0°C. and 760 millimeters pressure; inside gas is assumed to have everywhere the density corresponding to its unknown temperature and a pressure of 760 millimeters; and the outside air is assumed to have a uniform density corresponding to dry air at 0°C. and 760 millimeters pressure. Ans. 102°.5 C.

16. Plot a curve of which the abscissas represent hydrogen thermometer temperatures reckoned from ice point and of which the ordinates represent the differences between hydrogen thermometer temperatures and air thermometer temperatures. (b) Plot on the same sheet a curve of which the abscissas are hydrogen thermometer temperatures (reckoned from ice point) and of which the ordinates are the differences between hydrogen thermometer temperatures and mercury-in-glass temperatures.

17. A cheap thermometer is placed in a bath with a standard thermometer and simultaneous readings of the two are taken as the temperature of the bath is slowly increased giving the following results:

Cheap thermometer (Fahrenheit)	20°.0	32°.0	48°.0	60°.0
Standard (Fahrenheit)	21°.56	32°.81	49°.01	60°.80
Cheap thermometer	72°.0	88°.0	100°.0	116°.0
Standard	72°.50	88°.61	100°.49	116°.06

Plot a curve of which the abscissas represent the readings of the cheap thermometer and of which the ordinates represent the true corresponding temperatures.

Note. It is impracticable to make a standard thermometer so that its readings give true temperatures directly; the result of the careful standardization of a high-grade thermometer is a table of corrections from which the true temperature corresponding to any reading may be inferred. The readings of the standard thermometer which are given above are supposed to have been reduced in this way to true mercury-in-glass temperatures.

18. A steel meter scale is 99.981 centimeters long at 10°C. and 100.015 centimeters long at 40°C. At what temperature will the scale be exactly one meter long, assuming the expansion from 10°C. to 40°C. to be proportional to the increase of temperature? Ans. 26°.8.C.

19. A piece of soft wrought iron was found by Andrews to have a length of 101.5 centimeters at a temperature of 100°C. and a length of 101.77 centimeters at a temperature of 300°C. Find the mean coefficient of linear expansion of the iron between 100°C. and 300°C. Ans. 0.0000133.

Note. The mean coefficient of linear expansion between two temperatures is defined as the difference in length at the two temperatures divided by the length at the lower temperature and by the difference of temperature. From this definition we have

$$L_{t'} = L_t[1 + \alpha(t' - t)]$$

The value of α in this equation is slightly different from the value of α in the equation

$$L_t = L_0(1 + \alpha t)$$

but the difference is very small and is entirely negligible when one uses a tabulated value of a coefficient of linear expansion for purposes of calculation, unless the metal to which the calculation applies is known to be identically the same kind of metal as that for which the tabulated value of the coefficient was determined.

Different samples of commercial iron or steel or copper differ slightly in their expansion.

20. An iron steam-pipe is 1000 feet long at 0°C. and it ranges in temperature from − 20°C. to 115°C. What must be the range of motion of an expansion joint to provide for expansion? Ans. 1.539 feet.

Note. The coefficient of linear expansion of wrought iron for the given range of temperature is 0.0000114 according to Andrews.

21. A surveyor's steel tape is correct at 0°C. A distance as measured by the tape at 22°C. is 500 feet. What is the true value of the measured distance, coefficient of linear expansion of steel being 0.0000111? Ans. 500.1221 feet.

Note. The measured distance is 500 times as long as the portion of the tape between two adjacent foot-marks at the temperature at which the tape is used.

22. Ordinary steel rails 30 feet long are laid when the air temperature is 0.°C. What space must be left between the ends of the rails to allow for expansion, the maximum summer temperature of the rails being 50°C.? The coefficient of expansion of rail steel is 0.0000113. Ans. 0.017 feet.

Note. In the laying of steel rails provision is usually made for expansion. The rails of a long street car line are however frequently welded into one continuous piece of steel; and when such a rail cools to a low temperature it does not shorten but is thrown into a state of tension.

23. A steel bar of one inch section is stretched by an amount equal to 0.000226 of its length when subjected to a tension of 10,000 pounds. What tension would be required to keep this bar unchanged in length when it is cooled from 20°C. to − 10°C. Ans. 15,000 pounds.

24. Assuming the highest summer temperature as 45°C. and the lowest winter temperature as − 15°C., find the range of expansion of one of the 1700-foot spans of the Forth Bridge. The bridge is made of steel, the coefficient of linear expansion of which is about 0.0000113. Ans. 1.1526 feet.

25. A brass rod is 100 centimeters long at 10°C. and 100.171 centimeters long at 100°C. What is the mean coefficient of linear expansion of the brass for the given range of temperature? Ans. 0.000019.

26. A steel shaft is 20 inches in diameter at 70°F. A steel collar is to be shrunk upon this shaft. The collar is to be heated to 650°F. and have at that temperature an inside diameter of 20.01 inches, so that it may be easily slipped over the shaft. Required the inside diameter to which the collar must be turned in the shop, shop temperature being 70°F. The coefficient of linear expansion of steel is 0.0000113 per degree Centigrade. Ans. 19.937 inches.

27. A copper plate has an area of 20 square feet at 10°C. What is its area at 200°C.? The coefficient of linear expansion of copper is 0.000017 per degree Centigrade. Ans. 20.129 square feet.

28. A glass bottle is weighed as follows: (*a*) empty, 24.608 grams; (*b*) full of mercury at 0°C., 258.723 grams; and (*c*) full of mercury at 100°C., 255.133 grams. Find the coefficient of cubic expansion of the glass of which the bottle is made. Ans. 0.000027.

Note. The space inside of a vessel increases exactly as if it were a solid piece of the material of which the vessel is made. Therefore the mean coefficient of cubic expansion of the glass of which the bottle is made is equal to the difference in volume of the mercury in the bottle at 0°C. and 100°C. divided by the volume of the mercury at zero and by the difference of temperature.

CHAPTER II.

CALORIMETRY.

11. Complete statement of the first law of thermodynamics. A given substance is heated by the dissipation of work and brought back to its initial state by being cooled by contact with another (cooler) substance B. Then, if loss of heat to surrounding bodies is carefully avoided, the thermal effect produced in substance B is exactly the same as would be produced in it if it had been heated directly by the dissipation of the original amount of work. Therefore, *a substance which is heated by the dissipation of work stores something which is equivalent to the work and which is called heat.* The conception of heat as the energy of molecular motion is explained in Art. 2. Mechanical energy can be converted completely into heat but the conversion of heat into mechanical energy is subject to important limitations, as explained in Chapter V.

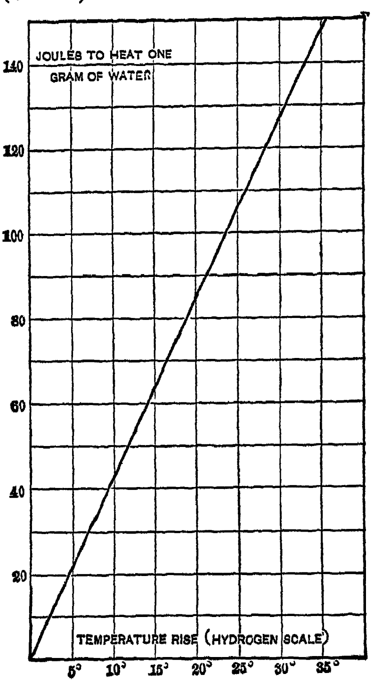


Fig. 8.

12. The heating of water by the dissipation of mechanical energy. The relation between the amount of work dissipated in heating water and the rise of temperature produced has been determined with great care. The results of Rowland's determination are given in the accompanying table and are shown

graphically in Fig. 8. The ordinates of the curve in Fig. 8 represent the amount of work in joules required to raise the temperature of one gram of water from 0°C. to t°C. (hydrogen scale). This same quantity of work is given in the table in the column headed E.

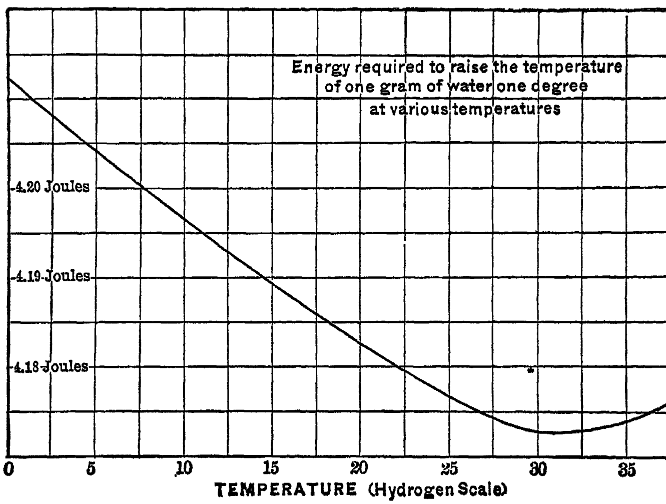


FIG. 9.

TABLE.*

Rowland's Determination of the Work Required to Heat Water.

Temperature rise from 0°C. to	Energy in joules to heat one gram of water.
(t)	[illegible]
5°	[illegible].040
10°	42.041
15°	63.005
20°	83.935
25°	104.834
30°	125.708
35°	146.745

The ordinates of the curve in Fig. 9 show Rowland's results in a slightly different form. For most practical purposes it is

*From Rowland's results by W. S. Day to the hydrogen scale. (*Physical Review*, Vol. VIII, April, 1898).

sufficiently exact to take 4.2 joules as the energy required to raise the temperature of one gram of water one degree centigrade (or 778 foot-pounds to raise the temperature of one pound of water one degree Fahrenheit).

Rowland's determinations were made by driving a rotatory paddle about a vertical axis at an observed speed in a vessel of water itself mounted on a vertical axis and prevented from turning by a cord passing over a pulley to a weight. The torque exerted by the paddle is thus equal to the product of the pull of the cord (which is equal to the attached weight) and the lever arm thereof. This torque multiplied by the angular velocity of the paddle in radians per second and by the time gives the work expended in heating the water. An accurate thermometer projecting through the hollow axle of the paddle indicates the rise of temperature of the water.*

13. Measurement of heat. An amount of heat, for example, the amount required to melt a gram of ice, or to raise the temperature of a gram of lead 1°C., is measured when the amount of work required to produce the effect has been determined. This measurement may be made by the direct determination of the work required to produce the given effect. The accomplishment of this method of heat measurement is, however, very tedious and subject to a very considerable error.† This is partly due to the difficulty of measuring work mechanically and partly due

*Using the data of Count Rumford's experiments (1798), one would find by calculation that about 847 foot-pounds are required to raise the temperature of one pound of water one degree. The first accurate and carefully planned determination was made by Joule, beginning in 1840. Joule's result was 772 foot-pounds per British thermal unit.

Rowland's experiments, which were carried out in 1879, are described in the *Proceedings of the American Academy of Arts and Sciences*, new series, Vol. 7. A good description of this work of Rowland's and of the work of other experimenters in the same field, is given on pages 267–286 of Edser's *Heat for Advanced Students*.

†The work spent in any portion of an electric circuit, can be measured with considerable accuracy and it can be easily applied to the accomplishment of any given thermal effect, and this electrical method for measuring heat values in energy units is perhaps the most accurate method at present available.

to the difficulty of applying mechanical work wholly to the heating of a given substance.

*Practical method of heat measurement. The water calorimeter.** The water calorimeter is a vessel containing a weighed quantity of water W arranged to absorb an amount of heat to be measured. Thus if the heat generated by the burning of a weighed quantity of coal is to be measured, the water calorimeter is arranged so that the whole of the heat generated by the burning of the coal may be absorbed by the water of the calorimeter. Figure 10 shows the water calorimeter as arranged for measuring the amount of heat given off by the cooling of a weighed amount of hot metal or other substance B. In this case the hot substance (at known initial temperature t) is plunged into the water of the calorimeter (at known initial temperature t'), the water is stirred vigorously by means of the stirrer SS, and the final temperature t'' of the water is observed. The quantity of heat which is absorbed by the water of the calorimeter is calculated as follows:

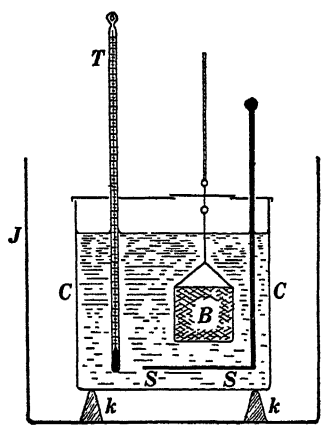


Fig. 10.

(*a*) *Accurate calculation.* Let E' be the energy in joules required to heat one gram of water from zero to t', and let E'' be the amount of energy in joules required to heat one gram of water from zero to t'' (see table on page 305). Then $W(E'' - E')$ is the amount of energy in joules required to heat the whole quantity of water in the calorimeter from t' to t'', where W is the mass of the water in grams. Therefore $W(E'' - E')$ is the energy value of the heat which has been imparted to the water of the calorimeter. It is, of course, necessary to make allowance for

*Other forms of calorimeter, such as the ice calorimeter and a variety of forms of the water calorimeter for special purposes are described on pages 117–163 of Edser's *Heat for Advanced Students*.

the amount of heat required to raise the temperature of the containing vessel of the calorimeter. This matter is explained below.

(*b*) *Approximate calculation.* The curve in Fig. 8 is plotted to represent accurately the tabulated values on page 305, and this curve is so nearly straight that it is impossible to represent it otherwise than by a straight line in so small a diagram as Fig. 8. That is to say, the energy required to heat a gram of water is approximately proportional to the rise of temperature, and *for most practical purposes the heat absorbed by a water calorimeter can be calculated by multiplying the weight of water in the calorimeter by the observed rise of temperature.* This gives the amount of heat in terms of what may be called the *water-unit of heat,* namely, the amount of heat required to raise the temperature of one gram of water one centigrade degree (the *calorie*), or the amount of heat required to raise the temperature of one pound of water one Fahrenheit degree (the *British thermal unit*).

The amount of heat required to raise the temperature of one gram of water one centigrade degree varies slightly with temperature, or in other words the curve in Fig. 8 is not a straight line The extent of this variation is shown in Fig. 9 which is derived from the results of Rowland's work on the heating of water as given in the table on page 305.

Calorimetric results may be expressed in energy-units or in water-units. It is desirable that all results of accurate calorimetric work be expressed in energy-units. It is usual, however, to express accurately measured heat values in terms of what is called the *standard calorie,* which is the amount of heat required to raise the temperature of one gram of water from 14°.5C. to 15°.5C. The standard calorie is equal to 4.189 joules. See Fig. 9.

The mechanical equivalent of heat, so called. In the early days of the nineteenth century heat was generally thought of as a peculiar kind of fluid, and the water-unit of heat was looked upon as a truly fundamental thing. So widespread was this point of view that Joule's result (1 British thermal unit = 778 foot-pounds) was for many years spoken of as the "mechanical equivalent of

heat"; the simple fact is, however, that by the **dissipation*** of 778 foot-pounds of work the temperature of one pound of water may be raised one Fahrenheit degree.

Sources of error in the use of the water calorimeter. (*a*) A portion of the heat imparted to the calorimeter is used to raise the temperature of the containing vessel, the stirrer and the thermometer bulb. The vessel and stirrer, being of the same material, are equivalent† to an amount km of water, where k is the specific heat (see Art. 14) of the material and m is the combined mass of vessel and stirrer. The water equivalent of the thermometer bulb is calculated in the same way, and in calculating the amount of heat absorbed by the calorimeter, the value $W + km + k'm'$ is used instead of W as the mass of the water in the calorimeter, km being the water equivalent of the vessel and stirrer and $k'm'$ being the water equivalent of the thermometer bulb.

(*b*) When the calorimeter is cooler than room·temperature it absorbs heat from its surroundings and *vice versa*. This source of error is to a great extent obviated by arranging that the initial temperature of the calorimeter may be as much below room temperature as the final temperature of the calorimeter is above room temperature, and by making the duration of the experiment as short as possible.

The vessel of the calorimeter CC, Fig. 10, should be made of thin polished metal and it should be surrounded by a polished metal jacket JJ with an air space between to reduce to a minimum the exchange of heat between the calorimeter and its surroundings. The calorimeter-vessel is usually supported on small wedges of cork.

(*c*) In a vessel of water which is being heated there are always large local differences of temperature. In order that the indica-

*See definition of the term dissipation of energy on page 279.

†An amount of metal and an amount of water are equivalent to each other in the sense here intended when a given amount of heat will produce the same rise of temperature of the one or of the other.

tions of the thermometer may be accurate, brisk stirring is necessary.

14. Specific heat-capacity of a substance.* The number of thermal units required to raise the temperature of one gram of a substance one centigrade degree is called the *specific heat-capacity* or the *specific heat* of the substance. Specific heats are ordinarily specified in calories per gram of substance per degree rise of temperature. It is sometimes desirable, however, to express specific heats in joules per gram per degree rise of temperature. Thus, the specific heat of water at 5°C. is about 4.204 joules per gram per degree centigrade (see Fig. 9).

The specific heats of most substances vary considerably with temperature. Thus, the ordinates of the curve in Fig. 9 show the specific heat of water at various temperatures expressed in joules per gram per degree centigrade. The *mean specific heat* of a substance for a given range of temperature t'' to t is the amount of heat required to heat one gram of the substance from t'' to t divided by $(t - t'')$, therefore the amount of heat required to heat S grams of the substance from t'' to t is equal to $kS(t - t'')$, where k is the mean specific heat of the substance, S is its mass in grams and $(t - t'')$ is the change of temperature. The mean specific heat of a substance is usually determined by heating S grams of the substance to a known temperature t and plunging it into a water calorimeter at known temperature t', the final temperature t'' of the mixture being observed. The amount of heat given off by the substance in cooling from t to t'' is $kS(t - t'')$, and this amount of heat, as measured by the rise of temperature of the water of the calorimeter, is equal to $W(t'' - t')$ as explained in Art. 13. Placing these two amounts of heat equal to each other, we have

$$k = \frac{W(t'' - t')}{S(t - t'')} \tag{10}$$

*The specific heats of a great variety of substances are given in John Castell-Evans' *Physico-Chemical Tables*, and in Landolt and Börnstein's *Physikalisch-Chemische Tabellen*.

in which W is the mass of water in the calorimeter (including the water equivalents of vessel and stirrer), S is the mass of the substance, t is the initial temperature of the substance, t' is the initial temperature of water in the calorimeter, and t'' is the final temperature of the mixture.

The extent to which the specific heat of a substance varies with temperature is further exemplified by Fig. 11. The ordinates of the curves in this figure show the specific heats of iron and the specific heats of crystalline carbon at various tempera-

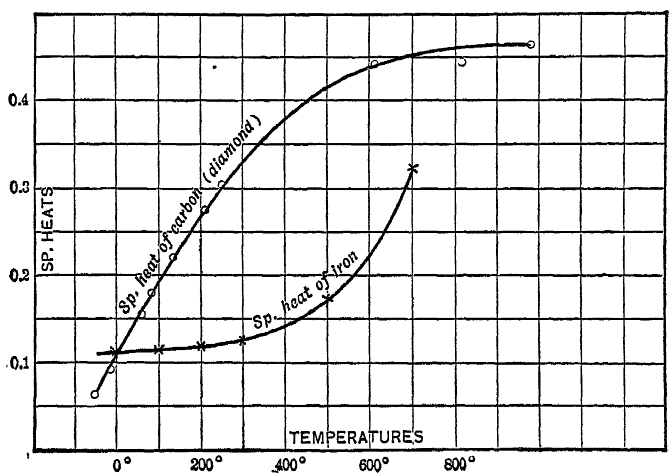


Fig. 11.

tures. As a rule the specific heat of a substance increases with rise of temperature.

15. Heat of combustion.* The number of thermal units of heat developed by the burning of unit mass of a fuel is called the *heat of combustion* of the fuel. For example, the burning of one gram of soft-wood charcoal generates 7070 calories.

Chemical action is in general accompanied by the development or the extinction of heat. Those chemical reactions during the

*Heats of combustion of a great variety of substances are given in Landolt and Börnstein's *Physikalisch-Chemicische Tabellen* and in John Castell-Evans' *Physico-Chemical Tables.*

progress of which heat is generated are called *exothermic reactions.* Those chemical reactions during the progress of which heat is absorbed or extinguished are called *endothermic reactions.* Combustion is the most familiar example of exothermic reaction. A very interesting feature of exothermic reactions is that such a reaction does not take place at an extremely high temperature. Thus hydrogen and oxygen exist side by side at extremely high temperatures without combining to form water vapor; on the other hand an endothermic reaction takes place eagerly (as it were) at extremely high temperatures. Thus at the temperature of the electric arc oxygen and nitrogen combine with absorption of heat. If we lived in an extremely high-temperature world we could cool our houses by the combustion of nitrogen and oxygen.

PROBLEMS.

29. (*a*) How many watts are required to raise the temperature of 10 liters of water from 0°C. to 10°C. in 20 minutes? (*b*) How many horse-power are required to raise the temperature of 10 gallons of water from 32°F. to 60°F. in 20 minutes? Ans. (*a*) 350 watts. (*b*) 2.76 horse-power.

Note. Solve this problem approximately by assuming that it takes 4.2 joules to raise the temperature of one gram of water 1°C. or 778 foot-pounds to raise the temperature of one pound of water 1°F. Solve the first part of the problem accurately by the use of the table on page 305.

30. Niagara Falls is 165 feet high. Calculate the rise of temperature of the water due to the energy of the fall on the assumption that the cooling effect of vaporization is zero. Ans. 0.212°F.

31. A copper calorimeter weighing 50 grams contains 500 grams of water at 16°C. A piece of copper weighing 65 grams is heated to 100°C. and plunged into the water of the calorimeter. The resulting temperature is 17°C. Find the specific heat of copper. Ans. 0.0935.

32. A piece of lead weighing 15 pounds at 100°C. is placed in a vessel containing 4 pounds of water at 15°C., and the temperature of the mixture finally settles to 24°.01C. The vessel is made

of copper and it weighs ½ pound. What is the specific heat of lead, the specific heat of copper being 0.093? Ans. 0.032.

33. A copper vessel weighing 2 kilograms contains 24 kilograms of water at 20°C. Into this vessel are dropped at the same instant, 2 kilograms of copper at 100°C., 2.4 kilograms of zinc at 60°C., and 6.4 kilograms of lead at 50°C. Find the resultant temperature. The specific heat of zinc is 0.093. Ans. 21°.2C.

Note. When there is no question as to the freezing of a portion of the water or the boiling of a portion of the water, the simplest argument of a problem of this kind is as follows: Let t be the resultant temperature and for the sake of argument let us think of t as being higher than any of the given temperatures. Then the specific heat of any one of the given substances multiplied by its mass and multiplied by (t minus initial temperature of substance) is the amount of heat required to raise the substance up to the resultant temperature. Adding all such products together gives the total heat required to raise the mixture up to its resultant temperature and this total heat is equal to zero.

34. A piece of steel weighing 6000 pounds is heated to a temperature of 1500°F. and plunged into an oil bath at 60°F. Assuming that all of the heat which is given off by the steel in being cooled to the final temperature of the bath is absorbed by the bath (no portion of it lost in vaporizing the oil), calculate the number of pounds of oil required in order that the final temperature of the bath may be 100°F. The specific heat of the oil is 0.7 and the mean specific heat of the iron between the specified temperatures is 0.2. Ans. 60,000 pounds.

35. A piece of platinum weighing 100 grams is drawn from a furnace and dropped into a water calorimeter at 20°C. and the resultant temperature is 59.2°C. The calorimeter vessel is of silver and weighs 15 grams (specific heat of silver 0.056) and the calorimeter contains 100 grams of water. The mean specific heat of platinum between ordinary room temperature and 1100° or 1200°C. is 0.0382. Find the temperature of the furnace. Ans. 1093°C.

36. The heat of combustion of hydrogen is 34,000 calories per gram. How many calories does this represent per gram of oxygen and how many calories per gram of water produced?

Ans. 4250 calories per gram of oxygen; 3777 calories per gram of water.

37. The heat of combustion of pure charcoal is 4000 British thermal units per pound when the product of the combustion is carbon monoxide CO and 14,500 British thermal units per pound when the product of the combustion is carbon dioxide CO_2. What is the heat of combustion of carbon monoxide per pound? Ans. 4500 British thermal units per pound.

38. Two volumes of carbon monoxide (CO) are mixed with 5 volumes of air in an explosion chamber, the temperature of the mixture being 20°C. and its pressure one atmosphere (14.7 pounds per square inch). Find the temperature and pressure immediately after the mixture is exploded by a spark. The heat of combustion of carbon monoxide is 2400 calories per gram of CO, the mean specific heat of carbon dioxide (CO_2) between the initial temperature and the temperature immediately after the explosion is 0.25 calories per gram and the mean specific heat of nitrogen for the same range of temperature is 0.3 calories per gram. Ans. 2428°C.; 116 pounds per square inch.
2435

Note. The specific heats above given are specific heats at constant volume.

Two volumes of carbon monoxide combine with one volume of oxygen to give two volumes of carbon dioxide so that 5 volumes of air contain the proper amount of oxygen to convert two volumes of carbon monoxide into carbon dioxide. The change of volume due to chemical combination is such that the 7 volumes of gas before the explosion would be reduced to 6 volumes after the explosion if the temperature and pressure were unchanged.

CHAPTER III.

THERMAL PROPERTIES OF SOLIDS, LIQUIDS AND GASES.

16. Melting points and boiling points.* When heat is imparted to a solid the temperature rises until the solid begins to melt; the temperature then remains constant until all of the substance is changed to liquid; the temperature then begins to rise again and continues to rise until the liquid boils; the temperature then remains constant until the liquid is entirely changed to vapor (pressure being unchanged); and then the temperature begins to

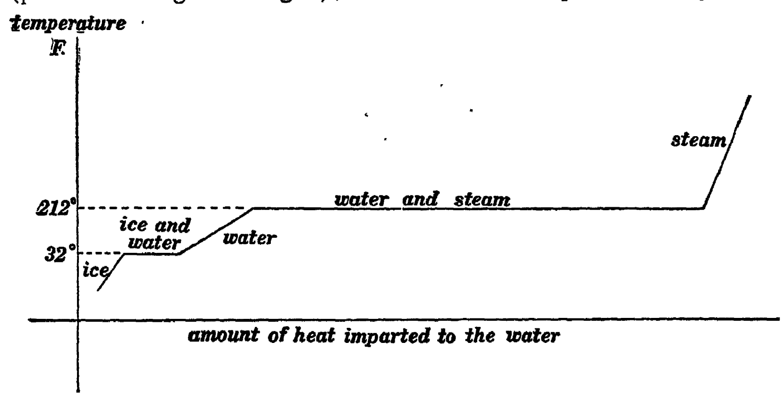


Fig. 12.

rise again as heat is continually imparted to the substance. There are thus two periods during which heat is imparted to a substance without producing rise of temperature, namely, when the substance is melting and when the substance is boiling under constant pressure.

Examples. Heat is imparted to very cold ice and the temperature rises until the ice begins to melt at 0°C., after the ice is all

*Extensive tables of melting points and boiling points are given in Castell-Evans' *Physico-Chemical Tables*, and in Landolt and Börnstein *Physikalisch-Chemische Tabellen.*

melted the temperature rises to 100°C., and after the water is all converted into steam the temperature of the steam is raised by further addition of heat, as shown by the ordinates to the curve in Fig. 12. The ordinates in Fig. 13 represent observed temperatures of a cooling crucible containing melted zinc. The temperature continues to fall until the zinc begins to freeze (at 418°), the temperature then remains constant until all of the zinc is frozen after which the temperature again falls.

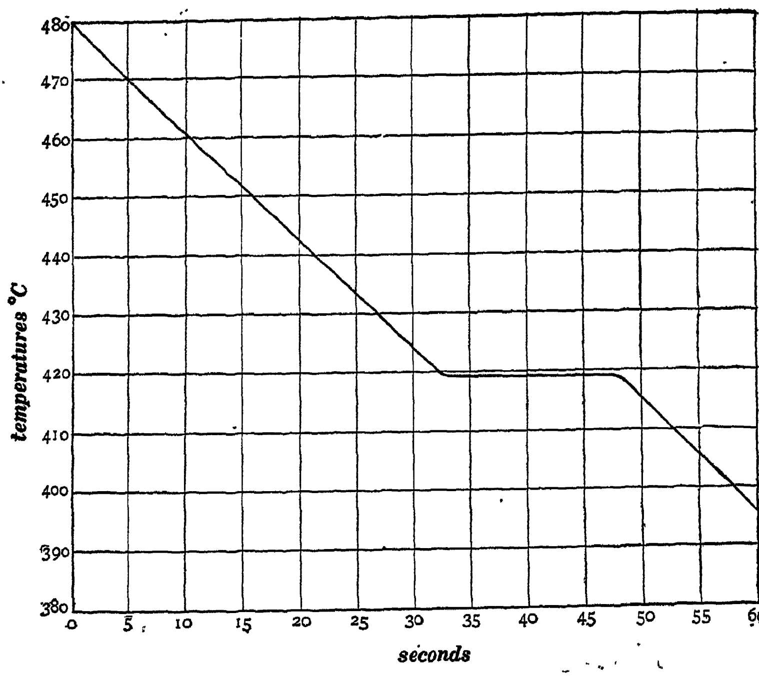


Fig. 13.

The *melting point* of a substance is the temperature at which the solid and liquid forms of the substance can exist together in thermal equilibrium. This temperature varies but slightly with pressure. All substances which like ice expand upon freezing have their melting points lowered by pressure, and all substances which like wax contract on freezing have their melting points raised by pressure.

The *boiling point* of a liquid at a given pressure is the temperature at which the liquid and its vapor can exist together in equilibrium. This matter is discussed more fully in the next article.

Every substance as far as known has a definite boiling point at any given pressure; but crystalline substances, only, have definite melting points. Thus, ice crystals and water exist side by side in equilibrium at a given temperature, whereas amorphous substances, such as glass and resin, have no definite melting point. These substances grow softer and softer with rise of temperature.

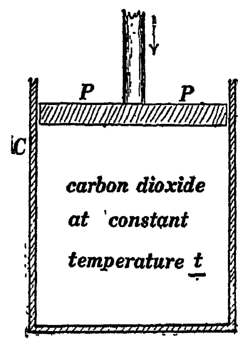


Fig. 14*a*.

17. Maximum pressure of a vapor at a given temperature and minimum temperature of a vapor at a given pressure. Consider a cylinder CC, Fig. 14*a*, filled with carbon dioxide and *kept at a fixed temperature t* while the piston PP is pushed downwards. At first the pressure of the gas increases as its volume decreases (in accordance with Boyle's Law), but when the pressure reaches a certain value (which depends upon the temperature t) then continued decrease of volume does not cause further increase of pressure but results in the condensation of a portion of the carbon dioxide into a liquid.* *There is thus a certain maximum pressure that gaseous carbon dioxide is capable of exerting at a given temperature and if an attempt is made to increase its pressure beyond this value by compression, a portion of the carbon dioxide condenses into liquid form, the remainder being at the same pressure as before.* The successive changes here described are represented graphically in Fig. 14*b*, in which abscissas represent volumes and ordinates represent pressures, everything being at a given temperature. The substance is wholly gaseous for the portion *a* of the curve, partly gaseous and partly liquid for the portion *b*, and wholly liquid for the portion *c*.

*If the temperature does not exceed a certain value which is called the *critical temperature* of the substance. See Art. 22.

The facts which are stated above and which are represented in Fig. 14*b* may be stated in a slightly different manner as follows; A cylinder *CC*, Fig. 15*a* is filled with carbon dioxide, and the

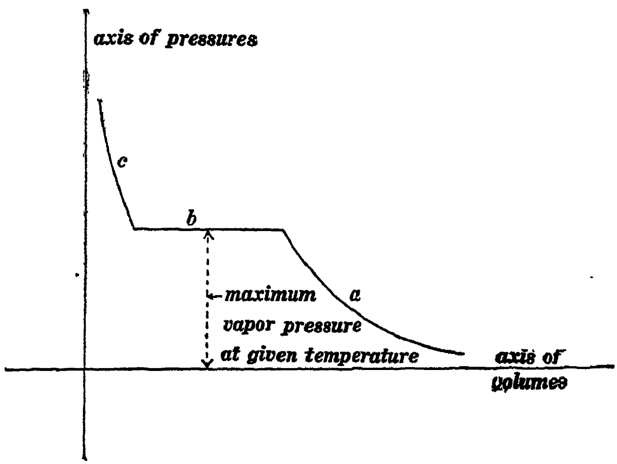


Fig. 14*b*.

temperature of the whole is slowly decreased, *the pressure of the carbon dioxide being kept at a constant value* by pushing the piston downwards if necessary. Under these conditions, the carbon dioxide remains wholly in a gaseous or vapor form until a certain temperature is reached below which the carbon dioxide cannot exist as a gas or vapor at the given pressure. When this temperature is reached, a continued abstraction of heat causes a portion of the carbon dioxide to condense to a liquid,* without decreasing its temperature, pressure being kept at a constant value. When, however, all of the carbon dioxide is condensed to liquid form, further abstraction of heat causes a further drop of temperature. The successive changes here described are

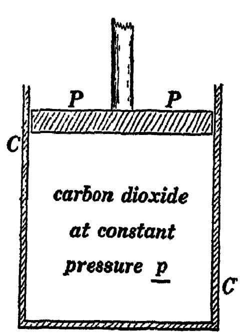


Fig. 15*a*.

*If the given pressure does not exceed what is called the *critical pressure* of the substance. See Art. 22.

represented graphically in Fig. 15*b*. For the portion *a* of the curve the carbon dioxide is wholly in the gaseous form. For the portion *b* it is partly gaseous and partly liquid, and for the portion *c* it is wholly liquid.

In discussing the change of a substance from a liquid to a gas or from a gas to a liquid it is customary to speak of the gaseous form of the substance as *vapor*. When the vapor is at its maxi-

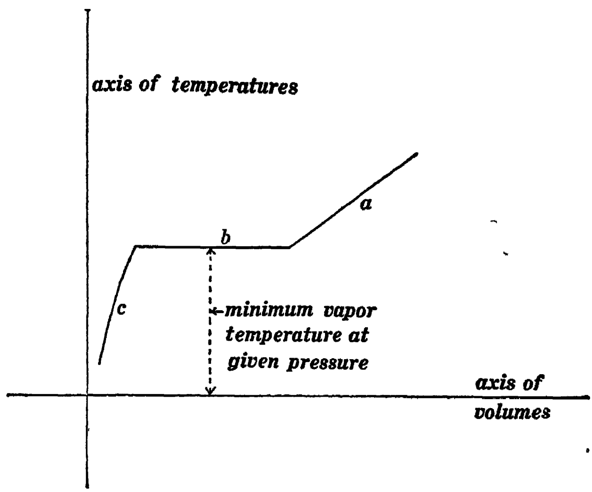


Fig. 15*b*.

mum pressure for a given temperature or at its minimum temperature for a given pressure, that is to say, when the pressure and temperature of the vapor are such that it would be in equilibrium with the liquid form of the substance if the liquid form were present, then the vapor is said to be a *saturated vapor*. A saturated vapor cannot be cooled without a portion of it being condensed if the pressure remains the same. A saturated vapor cannot be compressed without a portion of it being condensed if the temperature remains the same.

What is said here of carbon dioxide is true so far as known of every substance. Thus, water vapor at a given temperature cannot exert more than a certain maximum pressure, or at a given pressure it cannot be cooled below a certain minimum tem-

perature without condensation. The following tables give the maximum pressures and minimum temperatures of water vapor, and of anhydrous ammonia.*

TABLE.

Pressures and Temperatures of Saturated Water Vapor.

(Boiling Points of Water at Various Pressures).

Temp	Pressure in centimeters.	Temp	Pressure in centimeters.	Temp.	Pressure in centimeters.	Temp.	Pressure in centimeters.
−10°C.	0.2151 cm.	50°C.	9.1978 cm.	110°C.	107.537 cm.	170°C.	596.166 cm.
0°	0.4569	60°	14.8885	120°	149.128	180°	754.692
10°	0.6971	70°	23.3308	130°	203.028	190°	944.270
20°	1.7363	80°	35.4873	140°	271.763	200°	1168.896
30°	3.1510	90°	52.5468	150°	358.123	210°	1432.480
40°	5.4865	100°	76.0000	160°	465.162	230°	2092.640

TABLE.

Pressures and Temperatures of Saturated Ammonia Vapor (NH_3).

(Boiling Points of Liquid Ammonia at Various Pressures).

Temperature	Pressure in atmospheres.	Temperature	Pressure in atmospheres.
−30° C.	1.14 atm.	20° C.	8.41 atm.
−20°	1.83	40°	15.26
−10°	2.82	60°	25.63
0°	4.19	80°	40.59
10°	6.02	100°	61.32

The phenomenon of boiling.† According to the definition given in Art. 16, the boiling point of a substance at a given pressure is the minimum temperature of the vapor of the substance at that pressure. This temperature is perfectly definite at a given pressure. The temperature at which water boils, however, in the ordinary sense of that term (meaning the formation of bubbles of steam near the bottom of the vessel) is slightly variable, it depends to some extent upon the rapidity with which heat is given to the boiling water and to some extent upon cleanness of the vessel. The connection of the phenomenon of boiling with what is stated above concerning the minimum temperature of a vapor

*For more extensive tables see Castell-Evans' *Physico-Chemical Tables*, or Landolt & Börnstein's *Physikalisch-Chemische Tabellen.*

†See discussion of evaporation versus boiling in Art. 25.

at a given pressure may be explained as follows: Figure 16 represents a bubble of water vapor or steam S underneath water at atmospheric pressure. Therefore the steam itself must be at atmospheric pressure,* and if its temperature (the temperature of the water) is less than that for which steam can exert one atmosphere of pressure, the bubble of steam will condense into liquid and collapse. The temperature of the water must be at least as great as the minimum temperature of water vapor at atmospheric pressure in order that the bubble of steam may continue to exist, and if the temperature of the water is slightly greater than this the bubble of steam will continue to grow in size as more steam is formed at its boundaries.

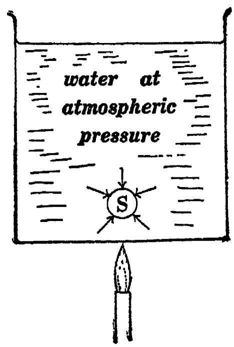


Fig. 16.

18. Variation of boiling and freezing points with pressure. As defined in Art. 16 the boiling point of a substance at a given pressure is the minimum temperature of the vapor of that substance at the given pressure. This temperature varies greatly with the pressure as exemplified in the above tables for water, and anhydrous ammonia. This variation of boiling point with pressure is utilized in the ammonia refrigerating machine. The essential features of this machine are shown in Fig. 17. Liquid ammonia vaporizes in the cooling pipes at a low pressure and low temperature, taking in heat from the surroundings. The pressure is maintained at a low value by means of a pump which removes the ammonia vapor from the cooling pipes and compresses it into condensing pipes as shown in the figure. The ammonia vapor is converted into liquid form in the condensing pipes, giving off heat to the surroundings, and the liquid ammonia is then allowed to flow back into the cooling pipes where it is again vaporized at low pressure and low temperature. If the

*A bubble of steam near the bottom of a deep vessel of boiling water is, of course, above atmospheric pressure.

pressure in the cooling pipes is kept at, say, 1.14 atmospheres the temperature of the boiling liquid ammonia will be about 30° below zero centigrade, according to the table on page

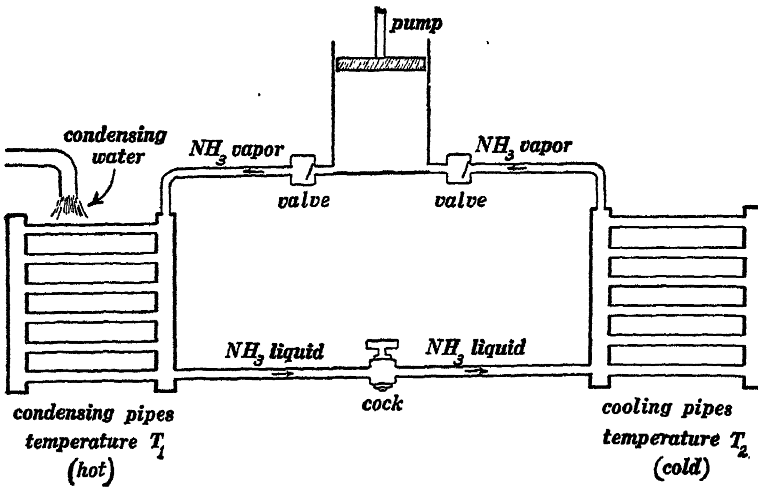


Fig. 17.

320, and if the pressure in the condensing pipes is kept at, say, 25.6 atmospheres, the vapor will condense at about 60° centigrade.

Regelation. The melting point of ice is lowered by pressure as stated in Art. 16. This is strikingly shown by the following experiment: A block of ice is supported on two pillars and a fine steel wire with a heavy weight on each end is hung over it. Where the wire rests against the ice the pressure is considerable and the melting point of the ice is low, say, one degree below zero centigrade. Some of the ice therefore immediately melts and causes * the temperature under the wire to fall to, say, one degree

*No solid can exist as a solid above its melting point. If one attempts to warm a solid above its melting point, a portion of the solid melts and the heat which is given to the substance goes to produce the change of state without causing a rise of temperature. When a block of ice at 0°C. is subjected to pressure, the melting point is lowered and the ice is therefore temporarily above its melting point. The result is that a portion of the ice melts and the heat used to produce the change of state cools the ice to its melting point.

below zero at which point the temperature under the wire stands as long as the pressure is maintained. The water from the melting ice flows around to the top of the wire where it is relieved of pressure and where it cannot, therefore, be cooler than 0°C. The result is that heat continually flows from the warm region (at 0°C.) on top of the wire into the cool region (at -1°C.) under the wire, the ice under the wire continues to melt, and the water as it flows around to the top of the wire continues to freeze. In the course of an hour or more a wire may thus cut its way through a large block of ice, leaving the block as one solid piece by the freezing of the water above the wire.

This melting of ice at a point where it is subjected to pressure and the immediate freezing of the resulting water when it flows out of the region of pressure, is called regelation. The remarkable ease with which a skater glides over the ice is due in large part to the formation of a thin layer of water in the region of excessive pressure under the skate runners. This water of course freezes almost instantly when the skate has passed and the pressure relieved. The cohesion of particles of ice when pressed tightly together, as exemplified in the packing of snow in balls, is due to the melting of the ice particles at the points of contact where the pressure is great, and the immediate freezing of the resulting water as it flows out of the small regions of pressure. Snow must be nearly at 0°C. in order that the phenomenon of regelation may be brought about by the slight pressure that can be produced by the hand.

19. Boiling points and melting points of mixtures. When a moderately dilute solution of common salt in water freezes, pure ice is formed, and the whole of the salt is left in the residual liquid. When a solution of salt in water boils, pure water vapor is formed, and the whole of the salt is left in the residual liquid. In every such case, namely, when the dissolved substance does not pass off with the water vapor or crystallize with the ice, the boiling point of the solution is higher than the boiling point of

pure water and the freezing point of the solution is lower than the freezing point of pure water.*

When a weak solution of salt in water is frozen, pure ice freezes out of the solution, the residual liquid becomes richer and richer in salt, and the freezing point decreases more and more until the residual liquid begins to freeze; and during the freezing of the residual liquid the freezing point does not change in value. When a very strong solution of salt in water is cooled, the solution "freezes" by the deposition of crystals of salt, the residual liquid becomes less and less rich in salt, and the freezing point lowers more and more until the residual liquid begins to freeze as before.

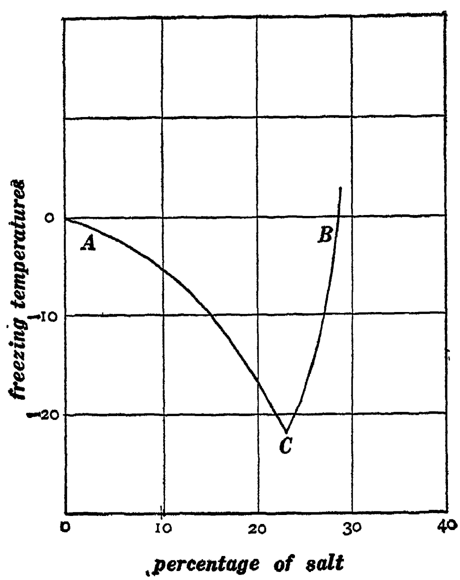


Fig. 18.

The ordinates of the curve *ABC*, Fig. 18, represent the freezing points of solutions of salt (NaCl) in water. At any point on the branch *AC* of the curve, the freezing consists in the deposition

*A good discussion of this subiect may be found in Whetham's *Theory of Solution*, Cambridge University Press, 1902. It is also discussed at considerable length in Nernst's *Theoretical Chemistry* (English translation Macmillan & Co., London, 1904), and in Jones's *Physical Chemistry* (The Macmillan Co., New York, 1902).

of ice crystals, and at any point on the branch *B C* the "freezing" consists in the deposition of crystals of salt. The point *C* of minimum freezing temperature is called the *eutectic point*, and the residual liquid which freezes at this temperature is called the *eutectic mixture* of salt and water. A microscopic examination of the frozen eutectic mixture of water and salt reveals the existence of minute crystals of pure ice and minute crystals of pure salt side by side.

The phenomena of freezing of fused mixtures of salts and the phenomena of freezing of metallic alloys are similar to the

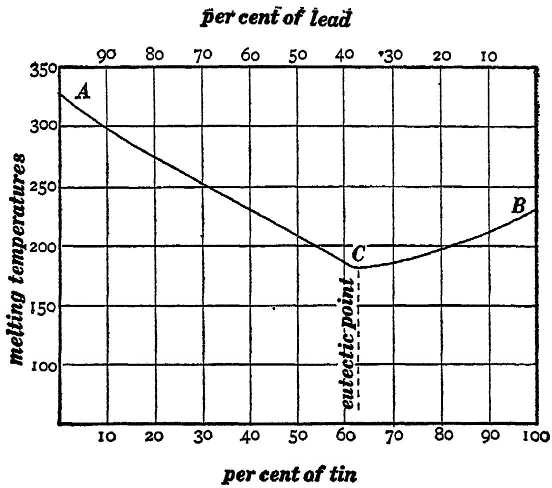


Fig. 19.

phenomena of freezing of salt solutions, and the above described behavior of a solution of common salt is perhaps the simplest case. Thus Heycock and Neville* have found many cases in which the freezing of a metallic alloy causes the deposition of pure crystals of one metal or the other, in the same way that the freezing of a salt solution causes the deposition of pure crystals of ice or pure crystals of salt. The freezing point of the residual alloy is steadily lowered by the deposition of pure crystals of

**Journal of Chemical Society* since 1888. See Nernst's *Theoretical Chemistry* (Macmillan & Co.), page 402.

either metal until a certain point (the eutectic point) is reached when the residual alloy (the eutectic alloy) continues to freeze without further drop of temperature.

Another case which is slightly more complicated is exemplified by alloys of lead and tin, of which the freezing-point diagram is shown in Fig. 19.* The alloy of the two metals which has the minimum freezing point is called the *eutectic alloy*. Along the branch AC of the curve, crystals of lead are deposited containing a variable percentage of tin ranging from pure lead at A up to 12 atoms of tin to 88 atoms of lead as the eutectic point C is approached; along the branch BC of the curve, crystals of tin are deposited containing a variable percentage of lead ranging from pure tin at B up to one atom of lead to 500 atoms of tin as the eutectic point is approached. The crystals of lead containing a variable percentage of tin and the crystals of tin containing a variable percentage of lead are called *solid solutions*.

Figure 20 is a melting-point diagram of alloys of copper and magnesium.† These alloys present three *eutectic points* as indicated in the figure and four so-called *distectic points*, or points of maximum freezing temperature. The first third of this diagram, between distectic points 1 and 2, is a melting-point diagram of mixtures of pure magnesium and the chemical compound Mg_2Cu; the middle portion of the diagram, between distectic points 2 and 3, is a melting-point diagram of mixtures of the two chemical compounds Mg_2Cu and $MgCu_2$; and the last third of the diagram, between distectic points 3 and 4, is a melting-point diagram of mixtures of the chemical compound $MgCu_2$ and pure copper.

When a cast metal is slowly cooled, the outside portions of the casting differ very considerably in composition from the interior portions of the casting; any substance which is present in the

*Taken from a paper by W. Rosenheim and P. A. Tucker, *Philosophical Transactions*, of the Royal Society, Series A, Vol. 209, pages 89–122, November 17, 1908.

†From a paper by G. Urazov, abstracted in the *Chemische Centralblatt*, page 1038 for the year 1908.

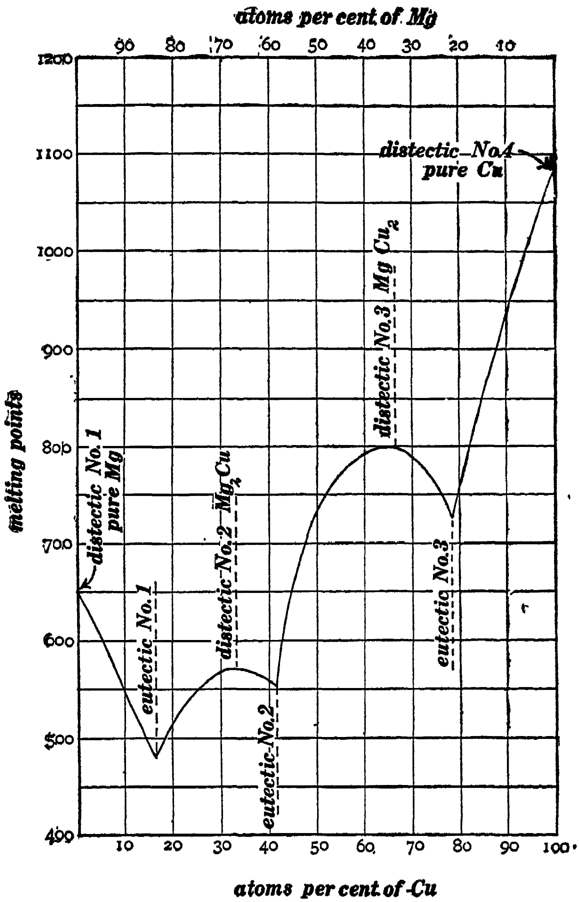


Fig. 20.

metal in small quantity tends to collect in the central parts of the casting.*

The use of ice and salt as a freezing mixture. Ice in a strong solution of common salt has a very low melting point, 15 or 20 degrees below zero centigrade (see Fig. 18). Therefore ice mixed with salt falls to a temperature of 15 or 20 degrees below zero centigrade, and stands at that temperature (if there is an excess

*A good introduction to the study of metallic alloys is Ibbotson's translation of Goerens' *Introduction to Metallography*, Longmans, Green & Co., 1908.

of undissolved salt) until all of the ice is melted by heat absorbed from surrounding objects. A vessel of pure water or cream surrounded by a mixture of ice and salt gives off heat to the very cold mixture until the water or cream is frozen. The sprinkling of salt on ice or snow in the winter time does not, as commonly supposed, melt the ice; it lowers the melting point below the temperature of the surroundings (if this is not more than 15 or 16 degrees below zero centigrade) and the ice is melted by the heat abstracted from its surroundings.

It should be especially noted that the action of the salt on the water is not the cause of the intense cold. Thus the combination of sulphuric acid and water causes a generation of heat and consequently a rise of temperature, whereas a mixture of ice and moderately dilute sulphuric acid falls to a very low temperature. The freezing point of the ice is lowered in the presence of acid or salt, some of the ice melts in consequence of this fact, and the heat which produces the melting is absorbed from the surrounding ice and solution, thus producing a fall of temperature.

Many substances cause a drop of temperature (an absorption of heat) when they are dissolved in water even when no ice is present. Thus every photographer is familiar with the very perceptible cooling which is produced when "hypo" (sodium hyposulphite) is dissolved in water.

20. Latent heat of fusion and latent heat of vaporization.* When heat is imparted to a substance which is at its melting point or at its boiling point, a portion of the substance is melted or vaporized and the temperature remains unchanged. The number of thermal units required to change unit mass of the solid substance at its melting point into liquid at the same temperature is called the *latent heat of fusion* of the substance. The number of thermal units required to change unit mass of a liquid † at its

*Berthelot's method for determining the latent heat of steam is quite fully described on pages 152 and 153 of Edser's *Heat for Advanced Students*. Methods for determining latent heats of fusion and latent heats of vaporization are discussed in every physical laboratory manual.

†In some cases the substance changes directly from the solid form to vapor without passing through the liquid state. This matter is discussed in Art. 23.

boiling point into vapor at the same temperature is called the *latent heat of vaporization* of the liquid.

The boiling point of a substance varies greatly with pressure, and the latent heat of vaporization varies greatly with the temperature of the boiling point of the given substance. Thus, the latent heat of vaporization of water is 1043 British thermal units per pound at a pressure of one pound per square inch (absolute) and at a temperature of 102°F., it is 965.7 British thermal units per pound at standard atmospheric pressure and 212°F., and it it 844.4 British thermal units per pound at 200 pounds per square inch (absolute) and 381°.6F.

The following table gives freezing and boiling points and latent heats of fusion and vaporization of a number of substances.

TABLE.

	Melting point.	Latent heat of fusion, calories per gram.	Boiling point at atmospheric pressure.	Latent heat of vaporization, calories per gram.
Water	0°C.	80	100°C.	536
Alcohol.........			78,3	209
Lead	327	5.86	1500. about	
Mercury........	−39.5	2.82	357.	62
Sulphur	115	9.36	444.7	
Ether			34.9	91
Carbon bisulphide			46.8	86.6
Chloroform			61.1	58.5

Total heat of steam. The amount of heat required to raise water from a chosen standard temperature, say 0°C., to the boiling temperature at a given pressure and to convert it into steam at that temperature and pressure is called the "*total heat*" *of the steam* at that temperature and pressure. According to Regnault's experiments, the total heat of steam is given with sufficient accuracy for most practical purposes by the equation

$$H=606.5+0.305t$$

in which H is the total number of calories required to raise the temperature of one gram of water from 0°C. to t°C. and convert it into steam at that temperature, the pressure of course being such as to cause the water to boil at temperature t.*

*Regnault's experiments are very briefly described in Edser's *Heat for Advanced Students*, pages 153–155. Very extensive tables of the total heat of steam are given in treatises on the steam engine.

21. Superheating* and undercooling of liquids. When water which is free from air and dust is heated in a clean glass vessel, its temperature is likely to rise 10 degrees or more above its boiling point (corresponding to the given pressure); and when it begins to boil it does so with almost explosive violence and the temperature quickly falls to the boiling point. If pure water is cooled in a clean glass vessel, its temperature is likely to fall considerably below its normal freezing point; and when freezing begins a large amount of ice is suddenly formed and the temperature quickly rises to the normal freezing point. It seems that water cannot change to vapor or to ice except there be some nucleus at which the change may begin. Most liquids show these phenomena of superheating and undercooling.

22. Critical states. When a liquid and its vapor (confined in a vessel) are heated, a portion of the liquid vaporizes, the pressure is increased, the density of the vapor increases and the density of the liquid decreases.† When a certain temperature is reached, the density of the liquid and the density of the vapor become equal and the vapor and liquid are identical in their physical properties. This temperature is called the *critical temperature* of the liquid, the corresponding pressure is called the *critical pressure*, and the corresponding density is called the *critical density*. The heat of vaporization of a liquid is less the higher the temperature (and pressure) at which vaporization takes place and it becomes zero at the critical temperature.‡

23. Sublimation. At a very low pressure the vapor of a given substance must be cooled to a very low temperature to produce

*The term *the superheating of a liquid* must not be confused with the term *the superheating of steam.* Superheated steam is unsaturated steam, that is, steam of which the pressure is less than the maximum pressure at the given temperature, or of which the temperature is greater than the minimum temperature for the given pressure. Steam may be superheated by passing saturated steam from a steam boiler through a coil of pipe in a furnace.

†This statement may not be exactly correct in some cases. The density of liquid and vapor become more and more nearly equal in every case.

‡A good discussion of the subject of critical temperatures and pressures including the celebrated experiments of Andrews on carbon dioxide is given in Edser's *Heat for Advanced Students*, pages 201–219. An introducing to van der Waal's theory of corresponding states is given in Edser's *Heat for Advanced Students*, pages 304–314. A very full discussion of van der Waals's theory of corresponding states is given in Nernst's *Theoretical Chemistry*, pages 224–230. Macmillan & Co., London, 1904.

condensation. If this temperature is below the freezing point of the substance, then the vapor will be condensed in solid form without passing through the intermediate liquid stage. Thus, to condense the slight amount of water vapor which is in the air in the winter time, the temperature of the air sinks far below the freezing point and the moisture is condensed in the form of snow-flakes or frost crystals. On the other hand, if the amount of moisture in the air on a cold winter's day is extremely small, snow and ice evaporate slowly without passing through the intermediate liquid stage. The word sublimation was applied by the early chemists to the process of distillation in which a solid is converted directly into a vapor and the vapor condensed into a solid. The most familiar example of this process of sublimation is that which is furnished by gum camphor. Every one perhaps has observed the formation of fine crystals of gum camphor on the cold side of a stoppered bottle which contains lumps of gum camphor. The gum camphor is converted into vapor in the warmer parts of the bottle and the vapor is condensed into crystals in the cooler parts of the bottle. The same phenomenon takes place in closed bottles containing crystals of iodine.

24. Pressures of mixed gases. When two or more gases are mixed in a vessel, the total pressure is equal to the sum of the pressures which each component gas would exert if it occupied the vessel alone (*Dalton's Law*). For example, if the amount of air in a vessel is such that it alone would exert a pressure p and if the amount of water vapor in the vessel is such that it alone would exert a pressure w, then the mixture will exert a pressure $p + w$.*

A result of Dalton's Law is that a definite portion of the total pressure of a mixed gas may be considered to be due to each of the component gases of which the mixture is made. Thus the total pressure of the atmosphere is due in part to the nitrogen, in part to the oxygen, in part to the carbon dioxide, in part to

*This statement is not exactly true, the degree of approximation being about the same as in the case of Boyle's Law and Gay Lussac's Law.

the water vapor, in part to the argon, etc., of which the atmosphere is a mixture.

25. Evaporation versus boiling. It is a common observation that water evaporates into the air at temperatures far below 100°C. *A liquid at a given temperature continues to evaporate so long as* **the pressure of its vapor is less than the maximum pressure its vapor can exert at the given temperature.** This is true whether the space above the liquid is filled with vapor alone or with vapor mixed with any gas at any pressure. For example, water vapor can exert a pressure of 355 millimeters of mercury at 80°C. and if a vessel at 80°C. contains water, the water will vaporize until *the pressure of the water vapor* in the vessel is 355 millimeters. If the vessel contains nothing but water vapor then, of course, the total pressure will be 355 millimeters when equilibrium is reached. If the vessel contains dry air at atmospheric pressure, some of the air will be driven out by the vapor which is formed, and when equilibrium is reached the water-vapor pressure in the vessel will be 355 millimeters and the air pressure will be 405 millimeters, making a total of 760 millimeters.* If the vessel is filled with dry air at any pressure p and suddenly closed before any perceptible amount of water vapor is formed, then water vapor will form until the total pressure is $p + 355$ millimeters, p being the pressure due to the air alone and 355 millimeters being the pressure of the water vapor.

26. Atmospheric moisture. Hygrometry. *Dew Point.* The dew point is the temperature to which the atmosphere must be cooled in order that the water vapor which is present may be saturated. Further cooling of the atmosphere would cause some of the moisture to condense.

Vapor pressure. That part of the pressure of the atmosphere which is due to the water vapor which is present is called the vapor pressure. This pressure varies from nearly zero to 30 millimeters, or more.

*The outside air pressure is assumed to be 760 millimeters.

Absolute humidity. The amount of water in the air, usually expressed in grams of water per cubic meter of air, is called the absolute humidity of the air. The absolute humidity varies from one gram of water, or less, per cubic meter of air on a very cold, dry winter's day to 30 or 35 grams of water per cubic meter of air on a moist summer's day.

Relative humidity. The amount of water in the air expressed in hundredths of what the air would contain if it were saturated at the given temperature is called the relative humidity. When the relative humidity is low, the air is said to be dry; when the relative humidity is high, the air is said to be moist, irrespective of the actual amount of water which is present. For example, 20 grams of water per cubic meter would correspond to a relative humidity of about 60 per cent. on a warm summer's day and the air would seem to be extremely dry, whereas about 5 grams of water per cubic meter of air would saturate the air at 0°C. and the air would seem extremely moist.

The method usually employed for the determination of the hygrometric elements (dew point, pressure of vapor, absolute humidity, and relative humidity) is by use of wet and dry bulb thermometers, from the readings of which the various quantities may be determined from empirical tables. Such tables are published by the United States Weather Bureau.*

27. Dissociation pressures. A phenomenon which is analogous to evaporation is the dissociation of a solid or a liquid substance by heat, a portion of the substance being given off in the form of vapor and the remainder being left in the form of a solid or liquid. An example will serve best to make the matter clear. Calcium carbonate (ordinary limestone) dissociates into calcium oxide (ordinary lime) and carbon dioxide when it is heated. If, however, calcium carbonate is brought to a given temperature in a closed vessel, the dissociation is arrested when the carbon dioxide reaches a definite pressure which is called the *dissociation pressure* of the calcium carbonate at the given tem-

*Weather Bureau Bulletin No. 235. Price, 10 cents.

perature. An increase of pressure or decrease of temperature casues some of the carbon dioxide to recombine with the free calcium oxide, and a decrease of pressure or increase of temperature causes some of the unchanged calcium carbonate to dissociate.

28. Transition temperatures. A crucible containing melted zinc is removed from a furnace and allowed to cool. The abscis-

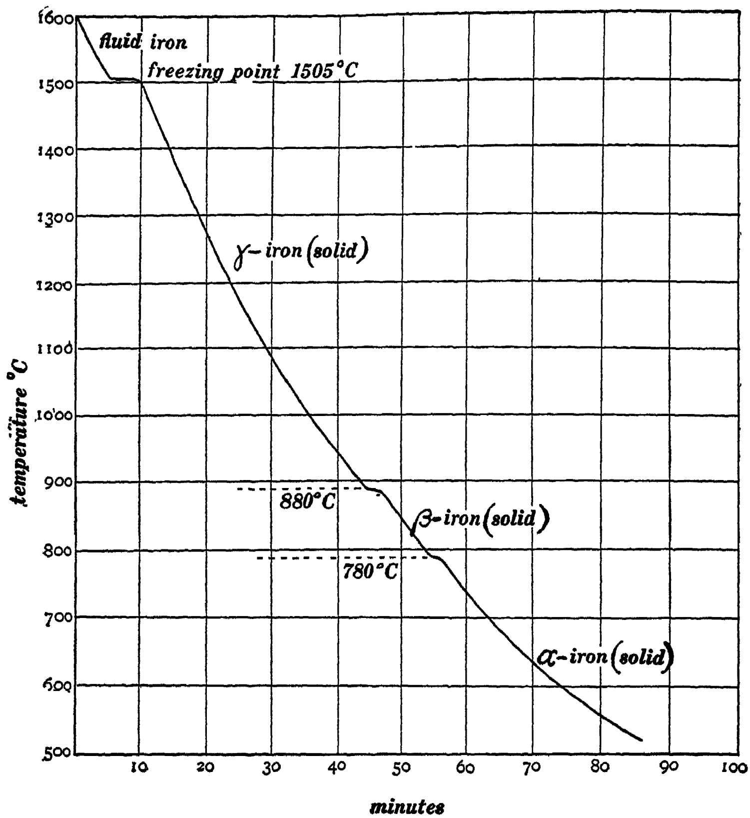


Fig. 21.

sas of the curve in Fig. 13 represent elapsed times and the ordinates represent observed temperatures of the zinc. The temperature of the metal drops steadily until it begins to freeze at

a temperature of 419°C., the temperature then remains constant until all of the zinc is frozen, after which the temperature again drops steadily. The curve in Fig. 13 is called a cooling curve.

Fig. 21 represents a cooling curve of a crucible containing pure melted iron. In this case the freezing of the iron is indicated

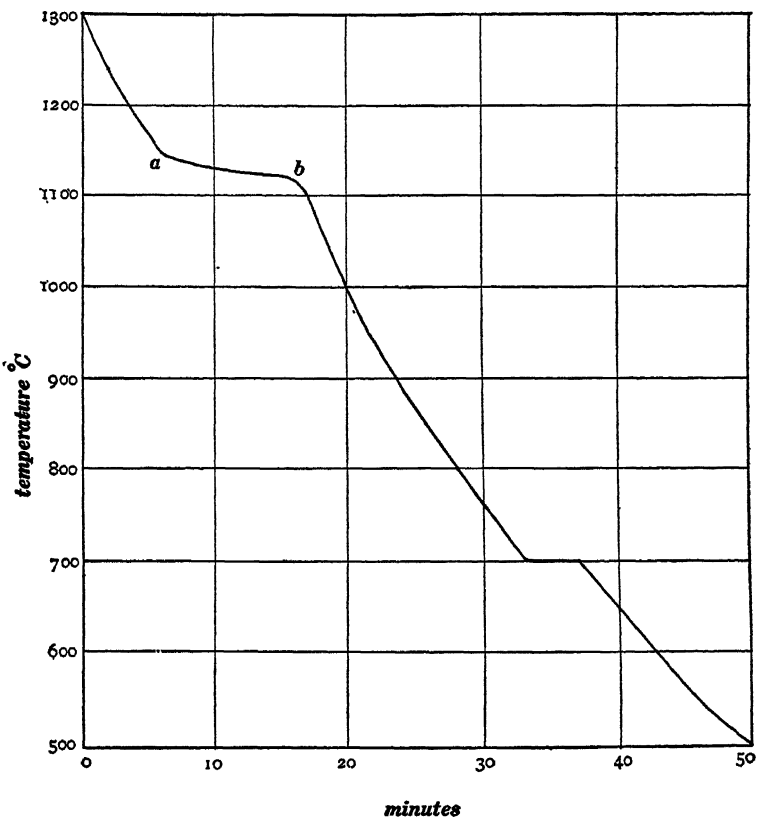


Fig. 22.

by a stationary temperature, and two other changes of state of the iron are indicated by stationary temperatures at 880°C. and 780°C. respectively. Solid iron seems to exist in three modifications which are called α-iron, β-iron and γ-iron, respectively. These modifications of solid iron differ in their crystalline structure. A remarkable physical property of α-iron is that it is

highly magnetic whereas β-iron and γ-iron are non-magnetic. The two temperatures 780°C. and 880°C. are called *transition temperatures* of iron.

The admixture of other substances affects not only the freezing point of a substance but also its transition temperatures. Thus cast iron is an alloy of iron with about five or six per cent. of carbon, and the cooling curve of cast iron is shown in Fig. 22. The freezing in this case begins at 1150°C. and the freezing point is steadily lowered to about 1120°C. as the freezing progresses.

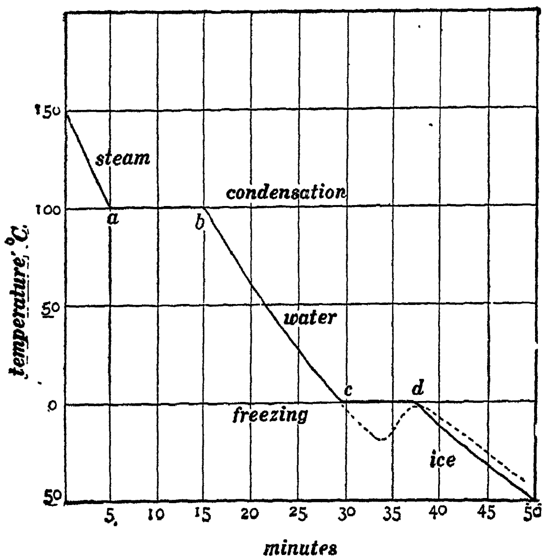


Fig. 23.

Below 1100°C. the iron is solid and the effect of the carbon is to lower both of the transition temperatures, which are shown in Fig. 21, to about 700°C. Thus cast iron has apparently but one transition temperature.

The recalescence of steel. The full-line curve in Fig. 23 is the cooling curve of water, the stationary temperature *ab* being that which occurs during condensation, and the stationary temperature *cd* being that which occurs during freezing. If the water is very pure it cools considerably below the freezing point

before freezing begins, as shown by the dotted line. When the temperature falls below the freezing point in this way a sudden rise of temperature takes place when freezing does begin, as shown by the dotted line. A phenomenon somewhat similar to this phenomenon of under-cooling of water occurs in cast iron and steel. When the cast iron or steel is cooled, the change from one modification to the other takes place after the temperature has fallen below the true transition temperature, and when the transformation does begin the temperature suddenly rises. This sudden rise of temperature is called *recalescence*, and it may be seen by heating a piano steel wire by an electric current and watching it as it cools; when it reaches a low red heat a sudden flash of brighter redness occurs after which the temperature again falls.

Retarded transformations. The hardening of steel. The phenomenon of the under-cooling of a liquid which is described in Art. 21 may be thought of as a slightly retarded transformation. The transformation from pure water to ice does not begin exactly at 0°C. when the water is cooled but at a lower temperature, and when the transformation does begin the mixture rises quickly to the normal freezing point. A familiar example of an almost permanently retarded transformation is that which is afforded by molasses candy. The crystallization of syrup is a very slow process because, apparently, of the viscosity of the syrup. If the syrup is cooled very, very slowly the crystallization takes place at the true freezing temperature, but if the syrup is cooled quickly it does not have time to crystallize, and the result is the well known molasses candy. Paradoxical as it may seem, the syrup when it is cooled suddenly does not have time to "freeze" but remains in that physical modification which is the stable modification at high temperatures. If, however, molasses candy is allowed to stand for some months the "freezing" gradually comes about, transforming the substance into the crystalline modification which is stable at low temperatures.

Retarded transformations take place in the hardening of steel and are of very great importance as follows. At a high temperature steel settles to thermal equilibrium with a certain crystalline structure, that is, with the iron in a certain modification and with certain crystalline compounds of iron and carbon present. If the steel is very slowly cooled the various transformations take place at approximately the true transition temperatures, and we have what is called *annealed steel* which is the stable form of steel at low temperatures. If, however, the steel is cooled very quickly the transformation from one modification to another does not have time to take place, the form or modification of the steel which normally exists and is stable at high temperatures is left in existence at ordinary temperatures, and we have the familiar hard form of steel. Hardened steel is an unstable modification and it tends gradually to change to the stable modification (soft annealed steel).* This change is greatly hastened by a slight rise of temperature. Thus hard steel is *tempered* by heating it slightly for a short time.

29. Coexistent phases. The forms of a substance which can exist together in thermal equilibrium are called coexistent phases of that substance. Thus the water in a vessel at a given temperature may be partly liquid and partly vapor (a liquid phase and a vapor phase); or the water in a vessel may be partly liquid and partly ice (a liquid phase and a solid phase). Water vapor may be wholly converted into water, and water may be wholly converted into ice, and vice versa; therefore these phases are said to be *phases of the same composition.*

A mixture of ice and salt in a vessel may be partly in the form of solution and partly in the from of crystals of salt, thus presenting a liquid phase and a solid phase. The liquid and the crystals in this case are *phases of different composition* inasmuch as a salt solution cannot be converted wholly into salt, nor can salt alone be converted into salt solution.

This breaking up of substances into phases of different composition is the fundamental fact of chemistry. Thus, when solutions of silver nitrate and sodium chloride are mixed, the mixture settles to thermal equilibrium with a solid phase consisting of precipitated silver chloride and a liquid phase consisting of a solution of sodium nitrate. When a mixture of alcohol and water is evaporated, the liquid and vapor both contain water and alcohol, but the percentage of alcohol is much greater in the vapor than in the liquid. Thus, if a weak solution of alcohol be partly

*Extremely hard steel gradually softens during the course of a number of years See a paper by Carl Barus. *Physical Review*, Vol. XXIX, pages 516–524, December, 1909.

distilled, the greater part of the alcohol passes over in the first distillate and the greater part of the water remains behind. This process is called *fractional distillation.*

30. Elementary and compound substances. A substance which can be broken up into phases of different composition is called a *compound substance.* Thus a salt solution is a compound substance and a mixture of alcohol and water is a compound substance because both can be separated into phases of different composition, water and salt in the one case and water and alcohol in the other case. The component parts of a given compound substance may themselves be compound. Thus, it is possible to separate water into phases of different composition. The simplest method for doing this is to pass an electric current through the water when two gases (oxygen and hydrogen) having entirely different properties are obtained. Substances which have never yet been broken up into phases of different composition are called *elementary substances* or *chemical elements.* For example, oxygen, hydrogen, iron, lead, sulphur, etc., are chemical elements.

31. Chemical compounds; mixtures. The component parts of some compound substances are always in unalterably fixed proportions. Thus, two volumes of hydrogen always combine with one volume of oxygen to produce water; 23.05 parts by weight of sodium always combine with 35.45 parts by weight of chlorine to form sodium chloride, and so on. Such compound substances are called *chemical compounds.* On the other hand, there are many compound substances of which the component parts may be in widely varying proportions; such compound substances are called *mixtures.* Thus, alcohol and water may be mixed in any proportion. Two substances are said to *combine* when they form a chemical compound and they are said to *mix* when they form a mixture. There is no very sharp line of division between these two processes, in general when substances combine (in fixed proportions to form a chemical compound) a large amount of heat is usually developed whereas when two substances mix (in variable proportions to form a mixture) a very small amount of heat is usually developed. The sharpest line of division between chemical compounds and mixtures, however, is that *a chemical compound freezes or evaporates at a fixed temperature,* pressure being constant. Thus, if water is cooled until it begins to freeze, the temperature will remain constant until it is all frozen, and if water is heated until it begins to boil, the temperature will remain constant until it is all vaporized. The freezing point of a mixture, however, slowly decreases as more and more of the mixture is frozen and the boiling point of the mixture slowly rises as more and more of the liquid is vaporized.*

32. Combining ratios. Law of constant proportions. Law of multiple proportions. Extremely fine iron filings may be burned in the air without any of the products of the combustion being scattered as is the case in the burning of coal. This burning of the iron filings is the combination of the iron with the oxygen of the air, the increase of weight after burning is the weight of the oxygen which has combined with the iron, and *the weight of the oxygen and the weight of the iron are in a fixed ratio* to each other. This is true of all chemical combinations, as stated in the previous article, and it is called *the law of constant proportions.*

*This statement is true in the great majority of cases, but it is not universally true.

The ratio of the weight of iron to the weight of oxygen which combines with it is called the *combining ratio* of these two elements. Two elements often have more than one combining ratio. Let b_1, b_2, or b_3, be the masses of one element which can combine with a given mass a of another element. *The masses b_1, b_2, b_3 are always multiples of some one number, that is to say, the ratios $b_1 : b_2$, $b_2 : b_3$,* etc., are *rational fractions.* This fact is called the *law of multiple proportions.* An illustration of this law is given in Art. 34 where the compounds of nitrogen and oxygen are described.

33. Chemical combination of gases. Let u and v be the respective volumes of two gases (measured at the same temperature and pressure) which unite to form a chemical compound. *The ratio u/v is always a rational fraction.**

Thus two volumes of hydrogen combine with one volume of oxygen to form water. Two volumes of carbon monoxide (CO) combine with one volume of oxygen to form CO_2, equal volumes of hydrogen and chlorine combine to form HCl. Nitrogen and oxygen may combine in the following proportions by volume: 2 of nitrogen with 1 of oxygen; 1 of nitrogen with 1 of oxygen; 2 of nitrogen with 3 of oxygen; 1 of nitrogen with 2 of oxygen; and 2 of nitrogen with 5 of oxygen.

34. The molecular theory. The above facts of chemical combination are clearly represented to our minds if we assume that each chemical element is made up of similar particles of equal mass called *atoms*, and that the atoms of two or more elements in a chemical compound are arranged in *similar groups* called *molecules*. For example, the atomic groups or molecules of the five compounds of nitrogen and oxygen are as follows:

Compound No. 1, 2 atoms of nitrogen and 1 atom of oxygen, N_2O
Compound No. 2, 1 atom of nitrogen and 1 atom of oxygen, NO.
Compound No. 3, 2 atoms of nitrogen and 3 atoms of oxygen, N_2O_3.
Compound No. 4, 1 atom of nitrogen and 2 atoms of oxygen, NO_2.
Compound No. 5, 2 atoms of nitrogen and 5 atoms of oxygen, N_2O_5.

The combining ratios of these various compounds are 2802 : 1588, 1401 : 1588, 2802 : 4764, 1401 : 3176, and 2802 : 7940, respectively. These numbers are multiples of 1401 on the one hand and of 1588 on the other hand. The number 1401 is called the atomic weight of nitrogen, and the number 1588 is called the atomic weight of oxygen.†

35. The principle of Avogadro. According to the molecular theory a definite number of atoms of one gas unite with a definite number of atoms of another gas to form a molecule when the gases combine chemically. When gases combine chemically, however, a definite number of volumes of the one gas always combine with a definite number of volumes of the other as explained in Art. 33. Therefore at an early stage of the development of the molecular theory, the hypothesis was advanced by Avogadro that *all gases have the same number of molecules per unit volume at the same temperature and pressure.* This hypothesis has been substantiated

*This statement is not exactly true. The degree of exactness is of the same order as the degree of exactness of Boyle's Law and Gay Lussac's Law.

†These are the atomic weights of nitrogen and oxygen when the atomic weight of hydrogen is arbitrarily chosen equal to 100.

by every bit of experimental evidence which has been brought to bear upon it; it is now considered to be established and it is called *Avogadro's principle.*

PROBLEMS.

39. A copper vessel weighing one kilogram contains 12 kilograms of water at 30°C. Into this vessel are dropped at the same instant one kilogram of copper at 100°C., 1.2 kilograms of zinc at 60°C. and 1.5 kilograms of ice at − 20°C. Find the resultant temperature. The specific heat of ice is 0.51 and the latent heat of fusion of ice is 80 calories per gram. Ans. 17°.65 C.

Note. The specific heat of copper and zinc are given in problems 32 and 33. The best method to adopt in the solution of such a problem as this is (1) to calculate the total amount of heat which would have to be taken from the mixture to bring everything to 0°C. If this amount of heat is less than enough to melt the given amount of ice, the fractional part of the ice which can be melted thereby can be calculated and in this case the resultant temperature is 0°C. with this fraction of the ice melted. If, however, the amount of heat which would have to be abstracted from the mixture to bring everything to zero is more than enough to melt all the ice then the amount required to melt the ice may be subtracted from the total amount and the rise in temperature produced in all the materials by the remainder of the heat may then be calculated.

40. An open vessel contains 500 grams of ice at a temperature of − 20°C. and heat is imparted to the vessel at the rate of 10 calories per minute. Plot a curve showing elapsed times as abscissas and temperatures of vessel as ordinates, assuming that the vessel gives no heat to surrounding bodies. The specific heat of ice is 0.51 and latent heat of fusion of ice is 80 calories per gram; the specific heat of steam is 0.38 and the latent heat of vaporization of steam at 100°C. and normal atmosphere pressure is 537 calories per gram.

41. Find the amount of heat required to raise 3 kilograms of lead at 10° to its melting point and melt it. The mean specific heat of lead between 10°C. and its melting point (325°C.) is about 0.035 and the latent heat of fusion of lead is 5.9 calories per gram. Ans. 50,775 calories.

42. How much water at 50°C. is required to melt 5 kilograms of ice at − 10°C.? Ans. 8.51 kilograms.

43. A gas is collected over water at a temperature of 15°C.

and the observed pressure is 752 millimeters. What would the pressure of the given amount of gas be if it occupied the same volume dry, that is, free from admixture of water vapor? Ans. 739.33 millimeters.

44. A gas is collected over water at a temperature of 18°C. The atmospheric pressure as determined by a barometer is 721 millimeters. The pressure of the gas in the vessel is less than atmospheric pressure by an amount which is equivalent to a column of water 104 millimeters high. The observed volume of the gas is 645 cubic centimeters. What volume would the gas have if measured dry at 0°C. and at a pressure of 760 millimeters? Ans. 555 cubic centimeters.

45. A closed bottle is full of dry air at 720 millimeters pressure and at a temperature of 50°C. A small quantity of water is introduced into the bottle and the whole is allowed to stand until the water vapor is saturated throughout the enclosed space. What is the total pressure of air and water vapor? Ans. 811.98 millimeters.

46. How much ice per day (24 hours) would be required to reduce from 25°C. to 17°C. an air blast which furnishes one cubic meter per second, the air being measured at 760 millimeters and at a temperature of 25°C.? Ans. 2028 kilograms.

Note. The density of air at 760 millimeters and 0°C. in 0.001293 grams per cubic centimeter, and the specific heat of air (at constant pressure) is 0.2375 calorie per gram.

47. The heat of combustion of good anthracite coal is 7800 calories per gram. A boiler is found by trial to evaporate 10 kilograms of water at 120°C. per kilogram of coal burned, the temperature of the feed water being at 20°C. Find the fractional part of the heat of combustion of the coal which is utilized in the boiler. Ans. 81.5 per cent.

Note. In this problem use the formula for total heat of steam on page 329.

CHAPTER IV.

THE ATOMIC THEORY OF GASES.*

(Units of the c.g.s. system are used throughout this chapter except where it is explicitly stated to the contrary, that is to say, pressure is expressed in dynes per square centimeter, volume in cubic centimeters, mass in grams, etc., and temperature is expressed in degrees centigrade. Absolute temperature is represented by T and temperature reckoned from the ice point is represented by t.)

36. The gas laws. The principle of Avogadro which was reached at the end of the preceding chapter represents the conclusion to which chemists have been led concerning the nature of a gas. This conclusion is that a gas consists of a great number of small particles and that at a given temperature and pressure all gases contain the same number of particles per unit volume. This hypothesis as to the atomic† character of a gas gives a clear insight into many of the physical properties of gases, and the object of the present chapter is to develop this aspect of the atomic theory. For purposes of ready reference the various experimental facts concerning gases are here collected.

Boyle's Law. The volume of a gas at constant temperature is inversely proportional to its pressure.

Gay Lussac's Law.‡ (*a*) The pressure of a constant volume of a gas is proportional to the absolute temperature.

(*b*) The volume of a given amount of gas at constant pressure is proportional to the absolute temperature.

General formula for Boyle's and Gay Lussac's Laws. The complete relation between pressure, volume and temperature of a gas is expressed by the formula

$$pv = MRT \tag{4 bis}$$

in which p is the pressure of the gas, v is its volume, M is its mass in grams, T is the absolute temperature, and R is a proportionality factor .

*What is here referred to as the atomic theory of gases is usually called the kinetic theory of gases.

†The terms *atom* and *molecule* refer to distinct ideas in chemistry. In physics, however, this difference is of little consequence except perhaps in some of the recent developments of spectrum analysis and in some of the recent studies of the discharge of electricity through gases. Therefore in this chapter the particles of a gas are called atoms or molecules indifferently.

‡The statements here given of Gay Lussac's Law involve the experimental fact which is stated in Art. 5 together with definition of temperature ratios as given in Art. 6. The statements here given are slightly misleading inasmuch as they make it appear that absolute temperature has been previously and independently defined.

Law of integral volumes. Let u and v be the respective volumes of two gases (reckoned at the same temperature and pressure) which combine chemically; then the ratio u/v is always a simple rational fraction.

Dalton's Law. A mixture of gases having no chemical action on each other exerts a pressure which is the sum of the pressures which would be exerted by each component gas separately if it occupied the containing vessel alone at the given temperature.

Joule and Thomson's principle. When a gas escapes through an orifice O, Fig. 24, from a region CC of high pressure into a region DD of low pressure, both pressures being kept at constant values by proper movements of the pistons A and B,

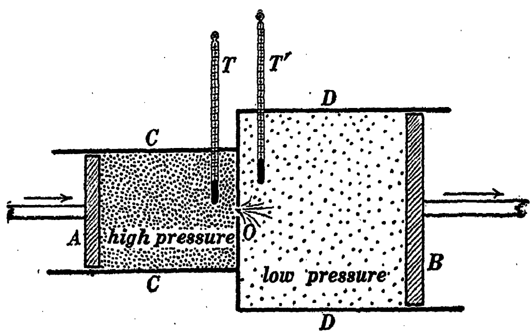


Fig. 24.

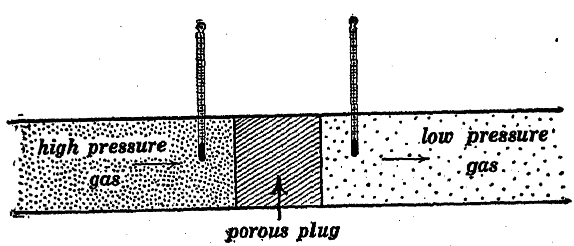


Fig. 25.

we have what is called *free expansion.* When a gas in a cylinder expands against a receding piston* we have what may be called *constrained expansion.* During constrained expansion (against a receding piston) a gas does work, as in a steam engine, the heat-energy contained in the gas decreases, and the temperature of the gas falls. The heat energy contained in a gas is not changed by free expansion, and the change of temperature as indicated by the thermometers T and T' in Fig. 24 during the free expansion of hydrogen, nitrogen or oxygen was found by Joule and Thomson (Lord Kelvin) to be very small at ordinary temperatures and pressures.

*It is important to notice that the moving piston A in Fig. 24 does not compress the gas in CC, it keeps its pressure and density constant; nor does the moving piston B allow the gas in DD to expand, it keeps its pressure and density constant.

Joule and Thomson's observations were carried on by means of the arrangement shown in Fig. 25. The innumerable small passages in a porous plug served instead of an orifice, and the difference of pressure on the two sides of the porous plug was maintained by means of a pump, the moving pistons of which take the place of A and B in Fig. 24.*

Joule and Thomson found that at ordinary temperatures and pressures hydrogen is slightly warmed by free expansion, whereas nitrogen and oxygen are slightly cooled. Gases having more complex molecules, for example, carbon dioxide (CO_2) and alcohol vapor (C_2H_6O) are cooled very considerably by free expansion. At very low temperatures and at very high pressures, all gases, as far as known, are very considerably cooled by free expansion. See Art. 38 on the liquid air machine.

37. The kinetic theory of gases. Imagine a large number of very small moving particles (molecules) enclosed in a vessel. Imagine these particles to have the property of rebounding with undiminished velocity when they strike the walls of the vessel, to be so small as seldom to collide against each other, and to exert no perceptible attraction or repulsion on each other. Such a system of particles would exhibit all the properties of a gas. Therefore a gas is thought to consist of such a system of particles or molecules.

Let p be the pressure of a gas, v the volume of the containing vessel (which is of course the volume of the gas), N the total number of molecules in the vessel, $n(=N/v)$ the number of particles per unit volume, and m the mass of each molecule. Now, the kinetic energy of the system of particles is constant since the particles rebound from the walls of the vessel with unchanged velocity. Therefore the *average* kinetic energy per molecule, namely, $\frac{1}{2}m\omega^2$, is constant and definite in value; the quantity ω^2 is the average value of the square of the velocities of the various particles; and we have:

$$p=\tfrac{1}{3}nm\omega^2 \tag{11}$$

*A good description of Joule and Thomson's experiments is given in Edser's *Heat for Advanced Students*, pages 379–392. A very complete discussion of the theory of Joule and Thomson's experiments, including a proper consideration of the work done on the gas by piston A in Fig. 24, and the work done by the gas on piston B, is to be found in Buckingham's *Thermodynamics*, pages 127–137 (The Macmillan Company, 1900). Buckingham's discussion also includes the application of the results of Joule and Thomson's experiments to the question of the interpretation of the indications of an air thermometer.

A discussion has arisen among physicists as to the cause of the decrease of temperature at the expansion nozzle in the liquid air machine (see Art. 38). This discussion has given rise to an experimental study of the subject by Bradley and Hale whose results are published in the *Physical Review*, Vol. 29, pages 258–292. The experimental work of Bradley and Hale is interesting and important but it was not needed to settle the question under dispute. The cooling effect is due to free expansion. Indeed, a cooling effect due to free expansion in Fig. 24 leads to a state of affairs in which *more work is done on the gas by piston A in Fig. 24 than is done by the gas on piston B* and this excess of work tends to cause a rise in temperature of the gas so that the actual observed difference of temperature on the thermometers T and T' in Fig. 24 is less than that which corresponds to free expansion alone.

or, since $n = N/v$,

$$pv = \tfrac{1}{3} Nm\omega^2 \tag{12}$$

Proof. The square of the velocity of a given particle is equal to the sum of the squares of the x, y and z components of its velocity. Therefore the sum of the squares of the velocities of all the particles is equal to the sum of the squares of all the x-components, *plus* the sum of the squares of all the y-components, *plus* the sum of the squares of all the z-components. The particles move at random in all directions, so that the sum of the squares of the x-components, of the y-components, and of the z-components are equal each to each. Therefore (i) *The sum of the squares, $N\omega^2$, of all the velocities is equal to three times the sum of the squares of the x-components.*

Imagine the containing vessel to consist of two parallel walls, of area q, distant d from each other, perpendicular to the x-axis of reference, and between which the gas is confined. Only the x-components of the molecular velocities contribute, by impact, to the pressure on these walls, so that the y and z components may be ignored. Consider a single particle, the x-component of whose velocity is a. This particle strikes first one wall and then the other, traveling back and forth $a/2d$ times per second. At each impact the velocity of the particle changes by $2a$, that is, from $+a$ to $-a$, or the momentum of the particle changes by $2am$. Therefore momentum is lost on each wall by the impact of this particle at the average rate $2am \times a/2d$, or ma^2/d, which is the average force exerted on the wall by this particle. That is, the force on one wall, due to one particle, is equal to m/d times the square of its x-velocity component. Therefore the total force F, exerted on the wall by all the particles, is equal to m/d times the sum of the squares of their x-velocity components. Therefore $F = \tfrac{1}{3}\omega^2 N \cdot m/d$; see ($i$). Dividing by q, and putting $qd = v$, we have $F/q = p = \tfrac{1}{3} Nm\omega^2/v$.

The kinetic theory of gases is very important as furnishing a clear conception of what constitutes thermal equilibrium of a gas, as furnishing a rational basis for Boyle's Law, Gay Lussac's Law, etc., and as enabling one to form clear mental pictures of various gas phenomena.

Thermal equilibrium of a gas. When a gas is in thermal equilibrium, the erratic movements of its molecules are such that on the average there is the same number of molecules in each unit of volume of the gas and the same average molecular velocity in the neighborhood of each point in the gas.*

Heating of a gas. When the walls of a containing vessel are heated the molecules of the enclosed gas rebound with increased velocity when they strike the walls and the temperature of the gas rises. When the walls of the containing vessel are cooled, the molecules of the gas rebound with diminished velocity when they strike the walls and the temperature of the gas falls.

*An interesting simple discussion of the kinetic theory of gases, including van der Waal's theory, is given in Edser's *Heat for Advanced Students*, pages 287–314. The kinetic theory of gases forms one of the most important parts of mathematical physics. A good elementary treatise on the subject is Boynton's *Kinetic Theory*, The Macmillan Co., New York, 1904. See also Boltzman's *Vorlesungen über Gas Theorie*. See also Planck, *Acht Vorlesungen über theoretische Physik* (Columbia University Lectures), Leipzig, 1910.

Heating of a gas by compression. Cooling of a gas by expansion. When a gas is compressed under a piston in a cylinder, the particles of the gas rebound from the inwardly moving piston with *unchanged velocity relative to the piston*, but with increased actual velocity, and the temperature of the gas rises. When a gas is expanded under a receding piston in a cylinder, the particles of the gas rebound from the receding piston with diminished actual velocity and the temperature of the gas falls.

Boyle's Law and Gay Lussac's Law. If we assume the absolute temperature of a gas to be proportional to the average kinetic energy per molecule, that is, if we assume T to be proportional to $\frac{1}{2}m\omega^2$, we may write *constant* $\times M \times T$ for $\frac{1}{3}Nm\omega^2$ in equation (12), and this equation then becomes

$$pv = MRT \tag{4)bis$$

in which R is a constant. On the basis, therefore, of the above assumption as to the relation between absolute temperature and average kinetic energy per molecule, the kinetic theory of gases is found to conform with Boyle's Law and Gay Lussac's Law.

Avogadro's principle is shown to be consistent with the kinetic theory of gases as follows: Consider two gases and let p_1, n_1, m_1, and ω_1 be the pressure, number of particles per cubic centimeter, etc., of the one gas, and let p_2, n_2, m_2, and ω_2 be the corresponding quantities for the other gas. Then $p_1 = \frac{1}{3}n_1m_1\omega_1^2$, and $p_2 = \frac{1}{3}n_2m_2\omega_2^2$, from equation (11). If the two gases are at the same pressure and temperature, then $p_1 = p_2$ and $m\omega_1^2$ must be equal to $m_2\omega_2^2$ according to the above assumption that absolute temperature is proportional to the average kinetic energy per molecule. Therefore when temperature and pressure are the same for the two gases, we have $n_1 = n_2$.

Dalton's Law is consistent with the kinetic theory of gases inasmuch as the moving particles are assumed to be so small that they do not interfere with each other in any way. Thus, if a number of oxygen molecules and a number of nitrogen molecules are in a containing vessel, each set of molecules will move exactly as if the others were not present and exert the same pressure as they would exert if they occupied the vessel alone.

Joule and Thomson's principle. Consider a molecule of gas as it darts through the orifice O in Fig. 24. This particle is moving away from the closely packed gas molecules in the high-pressure region and towards the more widely separated gas molecules in the low pressure region. Therefore, if there is an attractive force between the molecules of the gas, the molecule under consideration will lose velocity as it passes through the orifice because the backward force due to the attraction of the greater number of molecules behind it will exceed the forward force due to the attraction of the lesser number of molecules ahead of it. If, however, the molecules of the gas repel each other, a molecule will gain velocity when it darts through the orifice because the forward push of the greater number of molecules behind the given molecule will exceed the backward push of the lesser number of molecules ahead of it.

The very small change of temperature of hydrogen, nitrogen and oxygen during free expansion shows that the molecules of these gases do not attract each other or repel each other to any great extent; in fact hydrogen molecules at ordinary

pressures and temperatures (when the molecules are relatively far apart) repel each other slightly; nitrogen and oxygen molecules at ordinary pressures and temperatures attract each other slightly; and the molecules of all gases attract at low temperatures and pressures (when the molecules are relatively near together).

38. Equation of van der Waals.* All gases deviate more or less from the laws of Boyle and Gay Lussac and show perceptible change of temperature when allowed to expand freely through an orifice, and a mixture of two gases does not exert a pressure which is exactly equal to the sum of pressures which would be exerted by the respective components of the mixture if they occupied the entire containing vessel alone at the given temperature. The molecules are not so small as seldom to collide, and the molecules do in general attract or repel each other. When the effects of collision and of attraction (or repulsion) are taken into account in the kinetic theory, an equation between pressure, volume and temperature is deduced which is more complicated than equation (12). This equation is arrived at as follows, and it is due to van der Waals.

(*a*) *Effect of size of molecules.* If the moving particles have any size, collisions and impacts take place before the centers of the particles are coincident, or before the centers of the particles are in the plane of the wall of the containing vessel. Shorter distances are thus traversed between collisions, impacts are more frequent, and the pressure is greater than it would be if the particles were indefinitely small. The result is very much as if the volume of the containing vessel were smaller by a constant amount, b, than it really is. Equation (4b) may be modified so as to take account of this deviation, by writing $v-b$ for v. The value of b depends upon the amount and nature of the gas, and its value for one gram of a gas is called the *molecular volume* of the gas.

(*b*) *The effect of mutual attraction of particles* is to slow down the particles as they come into the layers of the gas adjacent to the walls. The attraction of the walls is constant and need not be considered. This slowing down of the particles makes the pressure of the gas less than it would otherwise be, by an amount which can be shown to be proportional to the square of the density of the gas or inversely proportional to the square of its volume. Equation (4b) may be modified, so as to take account of this deviation, by writing $p+a/v^2$ for p. The quantity a is a constant for a given amount of a given gas. Equation (4b) therefore becomes

$$\left(p+\frac{a}{v^2}\right)(v-b)=MRT \tag{13}$$

39. Linde's liquid air machine. At very low temperatures and at very high pressures the cooling effect of free expansion is very considerable and it is utilized in Linde's liquid air machine. This machine operates as follows: Air under great pressure (150 to 200 atmospheres) is forced through a large coil of small copper tube at the end of which it escapes through a fine orifice into a low pressure region, whence it flows back over the coil of copper tube. The expansion of the air at the fine orifice cools the air slightly; this slightly cooled air in flowing back over the coil of copper pipe cools the inflowing air, which in its turn is further cooled when it passes through

*A very complete discussion of gases, vapors, and liquids based upon van der Waals' equation is given in Nernst's *Theoretical Chemistry.*

the orifice; the inflowing air is then still further cooled in the coil of pipe, and so on, until the temperature at the orifice is so greatly reduced that a portion of the air condenses into a liquid which collects in the low-pressure chamber and is drawn off at will. If a liquid air machine were so arranged as to expand the high pressure air against a moving piston instead of allowing it to flow through an orifice, the doing of external work on the piston by the expanding gas would increase the cooling effect of the expansion and the efficiency of the liquid air machine would be increased.

40. The perfect gas. A beginner may obtain a good idea of the thermal properties of gases, sufficiently exact for all ordinary purposes, by assuming that Boyle's Law, Gay Lussac's Law, Joule and Thomson's principle, etc., are exactly true. All gases approach the ideal perfect gas at ordinary temperatures when highly rarefied, and the simple gases, that is to say, the gases of which the molecule is not complex, approximate very closely to the ideal perfect gas at ordinary temperatures and pressures. All gases depart widely from the ideal perfect gas at high pressures and low temperatures.

PROBLEMS.

48. Find the volume of 3.5 pounds of oxygen at a pressure of 3 atmospheres and at a temperature of 27°C., the volume of one pound of oxygen at 0°C. and one atmosphere being 11.204 cubic feet. Ans. 14.36 cubic feet.

49. One volume of oxygen, 2 atoms in the molecule, combines with 2 volumes of hydrogen, 2 atoms in the molecule, to form H_2O. How many volumes of H_2O vapor are produced? Ans. 2.

50. One volume of chlorine, 2 atoms in the molecule, combines with one volume of hydrogen, 2 atoms in the molecule, to form HCl. How many volumes of HCl gas are formed? Ans. 2.

51. Find the density in grams per cubic centimeter of a mixture of equal volumes of oxygen and hydrogen, the pressure of the mixture being 760 millimeters and the temperature of the mixture being 0°C., the density of oxygen at 0°C. and 760 millimeters being 0.00143 and the density of hydrogen at 0°C. and 760 millimeters being 0.000089 grams per cubic centimeter. Ans. 0.0007595.

Note. Each gas may be thought of as occupying the entire space alone at the given temperature and 380 millimeters pressure, and the density of each gas under these conditions calculated. The density of the mixed gas is then the sum of the densities of the two component gases.

52. Calculate the square root of the average square of the velocities of hydrogen molecules at 0°C.; and of oxygen molecules at 0°C.

Ans. Velocity of hydrogen molecules at 0°C. 184,200 centimeters per second. Velocity of oxygen molecules at 0°C. 46,050 centimeters per second.

Note. The product nm in equation (11) page 345 is the density of the gas, so that if the density and pressure of the gas are known the value of ω may be calculated. In fact one needs only to know the ratio of pressure divided by density, and this depends only on the temperature. At 760 millimeters pressure (= 1,013,000 dynes per square centimeter) the density of hydrogen at 0°C. is 0.00008954 gram per cubic centimeter and the density of oxygen is 16 times as great.

CHAPTER V.

THE SECOND LAW OF THERMODYNAMICS.*

41. A great deal of simple everyday knowledge is always taken for granted in a treatise on thermodynamics. In a previous article it was stated that the important things in connection with the generation of steam in a boiler by the burning of coal are: (*a*) the temperature of the feed water, (*b*) the temperature and pressure of the steam which is produced, (*c*) the character of the coal, (*b*) the temperature and composition of the air, and (*e*) the temperautre and composition of the flue gases. In a certain sense this is true, but, of course, the fundamentally important thing is the knowledge that coal will burn and convert water into steam. Such fundamental knowledge is, however, always taken for granted in the study of thermodynamics. The nature of fire is not an object of study in thermodynamics, but every one knows what fire is in a simple practical way; every one knows that an object becomes hot when it is placed on a hot stove; and every one knows that steam will squirt out of a hole in a steam boiler.

42. The subdivisions of physical science. The science of mechanics applies to the more or less ideal phenomena which are associated with the motion of rigid bodies either singly or in connected machines; with regular motion of distortion of elastic bodies like the bending of a bow or the oscillation of a string; and with ideally simply motion of flow of liquids and gases.† In every actual case of motion, however, we always encounter turbulence more or less marked, and the science of mechanics,

*Let the student remember that the term thermodynamics includes the whole of the theory of heat except the part which is based upon the atomic theory. Thus the preceding chapters, with the exception of Chapter IV, are chapters in thermodynamics.

†See Arts. 85 and 122 of *Mechanics*.

which is the science of describing the phenomena of motion, fails completely if we attempt to consider the minute details which are involved in this turbulence. Thus it is fairly easy to understand how the structural parts of a bridge stretch and shorten as a train passes across the bridge if one does not attempt to take account of the extremely complicated effects due to irregular gusts of wind, and to the swaying and rattling of the cars. Or consider the movement of the water at a certain point in a brook; there is indeed a fairly steady average velocity of the water at the point and a certain mean rhythmic variation, but superposed upon this average motion there is an erratic variation of velocity which is infinitely complicated.

Fire is the most familiar example of a turbulent phenomenon, and its most striking characteristic is that its progress is not dependent upon any external driving cause; when once started it goes forward of itself, and with a rush. Tyndall,* in referring to this matter says that to account for the propagation of fire was one of the philosophical difficulties of the eighteenth century. A spark was found sufficient to initiate a conflagration, and the philosophical difficulty lay in the fact that the effect seemed to be beyond all proportion greater than the cause. In discussing this matter Tyndall refers to Boscovich's explanation of the *sweeping* character of fire. He pictures a high mountain rising out of the sea with sides so steep that blocks of stone are just able to rest upon them without rolling down. He supposes such blocks, diminishing gradually in size, to be distributed over the mountain, large blocks below, moderate sized blocks at the middle height, and dwindling to grains of sand at the top. A small bird touches with its foot a grain at the summit; it moves, sets the next larger grains in motion, these again let loose the pebbles. these the larger stones, and these the blocks; until finally the whole mountain-side rolls violently into the sea.

The simple idea of cause and effect is a legitimate idea in the science of mechanics. Thus there is practically a definite rela-

*See Tyndall's *Heat a Mode of Motion*, page 66.

tionship between the amount of load on a bridge and the extent to which the bridge sags; but the simple idea of cause and effect cannot be applied to physical phenomena which involve turbulence. For example, the sun's rays heat the air next the ground over a large stretch of country, thus producing an unstable state of the atmosphere. Under these conditions, an extremely slight disturbance at one point may start the warm air moving upward, and from these slight beginnings a more and more violent chimney-like effect may develop, and lead to one of those great atmospheric movements which are called cyclones; and whether the cyclonic movement brings a severe storm to one or another part of the country may depend upon some insignificant character of the original infinitesimal disturbance, such as its place or time of occurrence.*

Every physical phenomenon involving turbulence is to some extent self-sustaining, every such phenomenon has a certain impetuous quality, and these remarkable characteristics of turbulence are now definitely formulated as the second law of thermodynamics.

The most important practical thing in connection with the turbulent aspect of any physical phenomenon is its general result or consequence, just as the important thing about the burning of a house is the loss. How utterly useless and uninteresting it would be, for example, to study the minutest details of a conflagration (assuming such study to be possible), recording the height and breadth and the irregular and evanescent distribution of temperature throughout each flicker of consuming flame, the story of each crackling sound and of every yield and sway of timber and wall! The fact is, we are immersed in an illimitable sea of phenomena every single detail of which is infinitely manifold, and no completely adequate science can ever be developed.

Physical science, aside from those branches which are depend-

*See a very brief article by W. S. Franklin in *Science*, Volume 14, pages 496–497, September 27, 1901.

ent upon the atomic theory, consists of three branches, namely: (1) *Mechanics*, including Hydraulics, Electricity and Magnetism, Light and Sound; the science of those phenomena in which turbulence may for practical purposes be ignored; (2) *Statistical Physics*, the science of those phenomena in which turbulence introduces an appreciable and practically important erratic element. Such phenomena can be studied only by the statistical method (the record of individual cases and the study of averages). Meteorology is the best example of statistical physics,* although every physical phenomenon has its statistical aspect; and (3) *Thermodynamics*. Some of the features of thermodynamics have already been pointed out. It is the study of changes of state of substances. A most important aspect of thermodynamics remains, however, to be considered, and a preliminary idea of this new aspect may be obtained by means of an analogy. In everyday life we see the fire-insurance companies concerned with certain broad features of statistical physics in their examinations and records of fires, and we see them also concerned with a profit and loss account which is wholly abstracted from the details of the phenomena of conflagration. Thermodynamics is the profit and loss branch of physics as it were; and like the profit and loss branch of fire insurance, thermodynamics is completely abstracted from any consideration of the details of any physical phenomenon. Thermodynamics is concerned with the measurement and accountting of that type of physical degeneration which accompanies turbulence just as fire insurance is concerned with the estimation and accounting of what we might call, using a fine phrase, structural degeneration by fire.

Thermodynamic degeneration. Every one has a feeling of the irreparable effects of disaster. The collapse of a bridge, the destruction of a house by fire or the wreck of a ship involve loss which indeed may be forgotten after reconstruction, but never balanced. The havoc that is wrought is essentially

*See two brief articles by W. S. Franklin, *Transactions of American Institute of Electrical Engineers*, Vol. 20, pages 285–286; and *Science*, Vol. 14, pages 496–497, September 27, 1901.

irreparable. It is desirable to use the word *degeneration* in a very narrow technical sense when we come to consider the second law of thermodynamics, and the way may be paved to a clear understanding of the later and accepted use of this word in physics by applying it now to designate that aspect of disaster which is irreparable. The burning of a building, for example, is a process of degeneration. The term *thermodynamic degeneration* applies to the effects of turbulence. Thus, a certain degeneration is associated with the turbulence which is produced when a hot iron is dipped into water, a certain degeneration is associated with the escape of a compressed gas through an orifice, a certain degeneration is associated with the flow of heat from a region of high temperature to a region of low temperature, a certain degeneration is associated with the conversion of work into heat by the rubbing of a coin on a board, and so on.

43. **Reversible processes.** A substance in thermal equilibrium is generally under the influence of external agencies. Thus surrounding substances confine a given substance to a certain region of space, and they exert upon the given substance a definite constant pressure; surrounding substances are at the same temperature as the given substance, and, according to the atomic theory, the molecules of the given substance rebound from surrounding substances with their motion on the average unchanged; surrounding substances may exert constant magnetic or electric influences upon the given substance; and so on. *If the external influences which act upon a fluid* in thermal equilibrium are made to change very slowly causing the pressure, volume, and temperature of the fluid to pass very slowly through a continuous series of values, and in general involving the doing of work upon or by the fluid, and the giving of heat to or taking of heat from the fluid, then the fluid will pass slowly through a process consisting of a continuous series of states of thermal equilibrium.* Such a process is called

*The thermodynamics of solids is extremely complicated except in a very few particulars. Therefore most of the general statements in the present discussion are limited to liquids and gases.

a *reversible process*, for the reason that the fluid will pass through the same series of states in reverse order if the external influences are changed slowly in a reversed sense. The characteristics of a reversible process are therefore as follows:

(*a*) A substance which undergoes a reversible process must be under varying external influences. A closed system* cannot perform a reversible process.

(*b*) A substance as it undergoes a reversible process is at each instant in a state of thermal equilibrium. If, at a given instant during a reversible process, the external influences should cease to change, no commotion would be left in the substance, and it would be on the instant in thermal equilibrium.

(*c*) A reversible process must take place slowly, indeed with infinite slowness. An actual process, that is a process which actually does proceed, can only be approximately reversible.

Examples of reversible processes. The very slow compression or expansion of a gas in a cylinder by the motion of a piston is a reversible process. The very slow heating of a gas in a closed vessel by placing the vessel in contact with a very slightly warmer substance is a reversible process because the gas is at each instant sensibly in thermal equilibrium.

44. Irreversible processes or sweeps. When a substance is settling or tending to settle to thermal equilibrium it may be said to undergo a process. Such a process cannot be arrested and held at any stage short of complete thermal equilibrium, but it always and inevitably proceeds towards that state. Such a process may, therefore, be called a *sweeping process* or simply a *sweep*. Consider, for example, the action which takes place when a red hot piece of iron is dropped into a pail of water. As the entire system settles to thermal equilibrium it passes through a series of stages each one of which grows out of the preceding stage inevitably, and it is impossible to arrest the process at any stage short of complete thermal equilibrium.

Molecular conception of a sweeping process. The molecular

*See *Mechanics*, Art. 58.

theory enables one to form a mental picture of a sweeping process. Thus Boscovich's idea of what we nowadays call an irreversible process or sweep is given in Art. 42 and it would be scarcely possible to improve on Boscovich's description in so far as the two most important characteristics of a sweeping process are concerned, namely, that a sweep is not dependent upon an external driving cause and that a sweep once started proceeds inevitably to a definite end.

45. Simple sweeps. The settling of a closed system to thermal equilibrium may be called a *simple sweep*. For example, the equilibrium of a mixture of oxygen and hydrogen in a closed vessel may be distributed by a minute spark, and the explosion of the gases together with the subsequent settling of the water vapor to a quiescent state constitutes a simple sweep. The equilibrium of a gas confined under high pressure in one half of a two-chambered vessel may be disturbed by opening a cock which connects the two chambers, and the rush of gas into the empty chamber constitutes a simple sweep.

46. Trailing sweeps. When external influences change continuously a substance in its tendency to settle to thermal equilibrium never catches up as it were with the changing conditions, but trails along behind them, and we have what may be called a *trailing sweep*. Thus, the rapid expansion or compression* of a gas in a cylinder is a trailing sweep. So long as the piston moves at a perceptible speed the gas in its tendency to settle to equilibrium never catches up with the varying conditions. This is evident when one considers that a sudden stopping of the piston would leave some slight turbulence in the gas which would not be the case if the gas were in equilibrium at the instant

*The different effects here mentioned in connection with rapid versus slow expansion must not be confused with the distinction between isothermic expansion and adiabatic expansion of gases. See Arts. 56 and 57. Isothermic expansion or compression must be slow in order that the necessary heat may be given to or taken from the gas to keep its temperature constant. On the other hand, any approximation to adiabatic expansion or compression of a gas in a cylinder must be rapid in order that there may be no appreciable amount of heat given to or taken from the gas by the cylinder walls.

the piston stopped. When the piston is moved more and more slowly, however, the departure of the gas from strict thermal equilibrium at each stage of the expansion or compression becomes less and less, and the expansion or compression approaches more and more nearly to a reversible process.

The rapid heating or cooling of a gas in a closed vessel is a trailing sweep. So long as heat is given to or taken from the gas at a perceptible rate there will be perceptible differences of temperature in different parts of the gas; and the gas in its tendency to settle to thermal equilibrium never catches up with the increasing or decreasing temperature of the walls of the containing vessel. When the gas is heated or cooled more and more slowly, that is, when heat is given to or taken from the gas at a rate which becomes more and more nearly imperceptible, then the departure of the gas from strict thermal equilibrium at each stage of the heating or cooling process becomes less and less and the heating or cooling process approaches more and more nearly to a reversible process.

47. Steady sweeps. A substance may be subjected to external action which although permanent or unvarying is incompatible with thermal equilibrium. When such is the case the substance settles to a permanent or unvarying state which is not a state of thermal equilibrium. Such a state of a substance may be called a *steady sweep*. For example, the two faces of a slab or the two ends of a wire may be kept permanently at different temperatures, and when this is done, the slab or wire settles to an unvarying state which is by no means a state of thermal equilibrium. Heat flows through the slab or along the wire from the region of high temperature to the region of low temperature, *never* from the region of low temperature to the region of high temperature. This flow of heat through the slab, or along the wire, is an irreversible process and it constitutes a steady sweep. The ends of a wire may be connected to a battery or dynamo so that a constant electric current flows through the wire, and the heat which is generated in the wire by the current may be steadily carried

away by a stream of water or air. Under these conditions the wire settles to an unvarying state which is by no means a state of thermal equilibrium, the battery or dynamo does work on the wire, and this work reappears steadily as heat in the wire. A reversal of the current *does not* reverse this process and cause heat energy to disappear in the wire (cooling of the wire) and reappear as work done in driving the dynamo as a motor or in recharging the battery; the process is irreversible and it constitutes a steady sweep.

48. Thermodynamic degeneration. Every one must admit that the impetuous character of a sweeping process suggests a certain havoc, a certain degeneration in the substance or system in which the sweep takes place. Consider, for example, a charge of gunpowder which has been exploded; if it is exploded in a large empty vessel, everything is there after the explosion, all of the energy is there and all of the material substance is there, but it cannot be exploded a second time.*

At this point of our discussion it is necessary to use the word *degeneration* so as to express more or less tentatively the idea that every sweeping process brings about a definite *amount* of degeneration, an amount that can be expressed numerically just as one speaks of so many pounds of sugar or so many yards of cloth. Thus, a certain amount of degeneration is brought about when a compressed gas escapes through an orifice, a certain amount of degeneration is brought about when heat flows from a region of high temperature to a region of low temperature, a certain amount of degeneration is brought about when work is converted into heat by friction or by the flow of an electric current through a wire, and so on.

*The man on the street has heard much during recent years of the conservation of energy and of the conservation of matter, and the old proverb that "you can't eat your cake and have it" presents to his mind a very simple fact concerning use and waste which in its less familiar aspects, as relating to engines for example, he tries in vain to rationalize in terms of these principles of conservation! Nearly all of the intuitive sense of the man on the street concerning use and waste (and he has a great deal) is involved in the second law of thermodynamics which is not a law of conservation at all. It is a law of waste.

In a simple sweep the degeneration lies wholly in the relation between initial and final states.* This is necessarily the case because no outside substance is affected in any way by the sweep, no work is done on or by the substance which undergoes the sweep and no heat is given to or taken from it. In a trailing sweep the degeneration may lie partly in the relation between the initial and final states of the substance which undergoes the sweep, partly in the conversion of work into heat and partly in the flow of heat from a high temperature region to a low temperature region. In a steady sweep, however, the substance which undergoes the sweep remains entirely unchanged as the sweep progresses, and *the degeneration lies wholly in the conversion of work into heat, in the transfer of heat from a region of high temperature to a region of low temperature, or in both.* Therefore the idea of thermodynamic degeneration as a measurable quantity can be reached in the simplest possible manner by a careful scrutiny of a steady sweep.

Proposition (a). *The thermodynamic degeneration which is represented by the direct conversion of work into heat* **at a given temperature is** *proportional to the quantity of work so converted.* Consider, for example, a steady flow of electric current through a wire from which the heat is abstracted so that the temperature remains constant. This process is steady, that is to say, it remains unchanged during successive intervals of time, and therefore any result of the process must be proportional to the time which elapses, that is to say, the amount of degeneration occurring in a given interval of time is proportional to the time, but the amount of work which is degenerated into heat is also proportional to the time. Therefore the amount of degeneration is proportional to the amount of work converted into heat at the given temperature.

Proposition (b). *The thermodynamic degeneration which is represented by the transfer of heat from a* **given** *high temperature* T_1

*The possibility of assigning a definite amount of thermodynamic degeneration to a given change of state of a substance depends upon Clausius's theorem which is discussed in Art. 54.

to a **given** *low temperature T_2 is proportional to the quantity of heat transferred.* Consider a steady flow of heat from temperature T_1 to temperature T_2 constituting a steady sweep, a sweep which remains entirely unchanged in character in successive intervals of time. Any result of this sweep must be proportional to the lapse of time, and therefore the degeneration which takes place in a given interval of time is proportional to the time; but the quantity of heat transferred is also proportional to the time, therefore the amount of degeneration is proportional to the quantity of heat transferred from temperature T_1 to temperature T_2.

Definition of the ratio of two temperatures. The definition of the ratio of two temperatures which is given in Art. 6 was understood to be a provisional definition. We are now in a position to propose a definition of the ratio of two temperatures which is independent of the physical properties of any particular substance. This definition will remain somewhat vague, however, until the action of the steam engine is discussed in the next article. According to proposition (*a*) above, the thermodynamic degeneration which is involved in the conversion of work into heat at a given temperature is proportional to the amount of work so converted, and the proportionality factor depends upon the temperature only. Therefore we may write

$$\varphi' = m_1 W \tag{i}$$

$$\varphi'' = m_2 W \tag{ii}$$

where φ' is the degeneration involved in the conversion of an amount of work W into heat at temperature T_1, φ'' is the degeneration involved in the conversion of an amount of work W into heat at temperature T_2, and m_1 and m_2 are factors which depend upon T_1 and T_2 respectively.

The amount of work W having been converted into heat at temperature T_1, imagine the heat to flow to a lower temperature T_2, thus involving an additional amount of degeneration accord-

ing to proposition (*b*) above. The conversion of work W into heat at temperature T_1 and the subsequent flow of this heat to a lower temperature T_2 gives the same result as would be produced by the conversion of the work into heat at the lower temperature directly. Therefore, the lower the temperature at which work is converted into heat the greater the amount of degeneration involved. That is to say, the factor m_2 in equation (ii) is larger in value than the factor m_1 in equation (i), temperature T_1 being higher* than temperature T_2. Therefore, since m_1 and m_2 depend only upon T_1 and T_2 respectively, it is permissible to adopt the equation

$$\frac{T_1}{T_2} = \frac{m_2}{m_1} \tag{iii}$$

as the definition of the ratio T_1/T_2. This definition of temperature ratios will be explained more in detail in the next article. The definition is due originally to Lord Kelvin.

Another way to express the definition which is involved in equation (iii) is to consider that the factor m_1 is the smaller the higher the temperature T_1, so that we may adopt k/m_1 as the measure of the temperature T_1, and k/m_2 as the measure of the temperature T_2, giving

$$m_1 = \frac{k}{T_1} \tag{iv}$$

and

$$m_2 = \frac{k}{T_2} \tag{v}$$

where k is an indeterminate constant. Therefore equation (i) and (ii) may be written in the general form

$$\varphi = \frac{kW}{T} \tag{vi}$$

*The idea of higher and lower temperature is not dependent upon any method of measuring temperature. When a substance receives heat definite observable effects are produced, and when these effects are produced by placing one substance into contact with another substance, the other substance is known to give heat to the given substance and its temperature is known to be higher than the temperature

where φ is the thermodynamic degeneration involved in the conversion of an amount of work W into heat at temperature T, and k is an indeterminate constant.

*The ratio of two temperatures as defined by equation (iii) is very nearly the same as the ratio of two temperatures as measured by the gas thermometer, and therefore gas thermometer temperatures (so called absolute temperatures as measured by the gas thermometer) may be used throughout this chapter without appreciable error.** Now, since the factor k in equation (vi) is indeterminate we may choose as our unit of thermodynamic degeneration the amount which is involved in the conversion of one unit of work into heat at a temperature of one degree on the "absolute" scale; then the value of k is unity and equation (vi) becomes

$$\varphi = \frac{W}{T} \tag{14}$$

When W is expressed in joules and T in degrees centigrade, φ is expressed in terms of joules per degree. Thus one joule per degree is the degeneration involved in the conversion of one joule of work into heat at 1°C. absolute, or the amount involved in the conversion of 1000 joules into heat at 1000°C. absolute.

To convert an amount of work W into heat at temperature T_1 involves W/T_1 units of degeneration, to convert the same amount of work into heat at temperature T_2 involves W/T_2

of the given substance. Thus a piece of wax is melted when it is placed on a hot stove.

*If the gas used in the gas thermometer conforms exactly to Boyle's Law and if the temperature of the gas would neither rise nor fall during free expansion, then absolute temperatures as measured by the gas thermometer would coincide exactly with temperatures as defined by equations (iv) and (v). This proposition is established in Art. 58 on the basis of the theory of the perfect engine as outlined in Art. 50.

If the gas used in the gas thermometer does not conform to Boyle's Law (and no gas does conform to Boyle's Law rigorously) and if the gas does change its temperature during free expansion (and all gases do more or less), then temperatures as measured by the gas thermometer depart slightly from temperatures as defined by equations (iv) and (v). This matter is discussed fully in Buckingham's *Thermodynamics*, pages 127–136 (The Macmillan Company, New York, 1900).

units of degeneration, and therefore to transfer an amount of heat equal to W from temperature T_1 to temperature T_2 must involve an amount of degeneration equal to the excess of W/T_2 over W/T_1 or an amount equal to

$$W\left(\frac{1}{T_2}-\frac{1}{T_1}\right) \quad \text{or} \quad H\left(\frac{1}{T_2}-\frac{1}{T_1}\right)$$

where H is the amount of heat transferred.

49. The second law of thermodynamics. (*a*) *The thermodynamic degeneration which accompanies an irreversible process cannot be directly repaired, nor can it be repaired by any means without compensation.* This is an entirely general statement of the second law of thermodynamics. The *direct repair* of the degeneration due to a sweeping process means the undoing of the havoc wrought by the process by allowing the sweeping process to perform itself backwards, an idea which is exactly as absurd as the idea of allowing a burned house to unburn itself. Following are several specialized statements of the second law of thermodynamics.

(*b*) *Heat cannot pass directly from a cold body to a hot body, nor can heat be transferred from a cold body to a hot body by any means without compensation.*

(*c*) *Heat cannot be converted directly into work, nor can heat be converted into work by any means without compensation.*

The *direct conversion* of heat into work would be the simple reverse of any of the ordinary sweeping processes which involve the degeneration of work into heat (direct conversion of heat into work would be to allow the sweeping process to perform itself backwards). For example, work is degenerated into heat in the bearing of a rotating shaft, and we all know that to reverse the motion of the shaft does not cause the bearing to grow cold and the heat so lost to appear as work helping to drive the shaft. That would be a rotary engine indeed! There is an important general theorem in thermodynamics to the effect that if two sweeping processes A and B involve the same amount of de-

generation, and if either of the processes, say A, has been allowed to perform its sweep, then by a lever arrangement, as it were, the process B can be carefully *let down*, and the havoc wrought by the sweep of A can be undone. The result of this operation, however, would be to leave the system B in the condition into which it would be degenerated if the process B had been allowed to sweep instead of being *let down*. This is very much as if, having two similar houses A and B, one of which A has been burned, we could rig up a mechanism which would *let down* B to ashes and cause A to be restored in the original actual materials of which it was first constructed. This is of course impossible in the case of the two houses but it is possible in every known case of thermodynamic degeneration. This general theorem is as thoroughly established as any generalization in physics, and if it is true and if we ever find a way to convert heat into work unconditionally and without cost or compensation, then it will be proven indirectly that a shaft *can* be driven by heating one of the bearings in which it rotates, for direct conversion of heat into work by one process must be according to this general theorem equivalent to and replaceable by the reverse of any ordinary sweeping process which converts work into heat.

(*d*) *A gas cannot pass directly from a region of low pressure to a region of high pressure, nor can a gas be transferred from a region of low pressure to a region of high pressure by any means without compensation.*

Imagine a gas squirting itself backwards through a nozzle into a high pressure reservoir! The repeated statement of self-evident facts concerning *direct repair* in these statements of the second law of thermodynamics may seem ridiculous to the intelligent reader, but the second law of thermodynamics is a statement of a fact which every one knows coupled with a generalizing clause which once thoroughly understood is almost if not quite self-evident.*

*Here is one more statement of the second law of thermodynamics, the oldest English version of it:

Let us return to the fourth statement (*d*) and consider with the help of an example what is meant by compensation in its thermodynamic sense. A gas *can* be transferred from a region of low pressure to a region of high pressure by means of a pump, and the work that is done in driving the pump, even supposing the pump to be frictionless, is all converted into heat. This conversion of work into heat is the necessary cost or compensation for the transfer of the gas from a low pressure region to a high pressure region.

Consider the second statement (*b*); in an artificial-ice factory heat *is* continually abstracted from the freezing room and transferred to the warm outside air; but to accomplish this result, even by an ideally perfect, frictionless mechanism, a certain amount of work is required to drive the ammonia pump and this work is converted into heat (the amount of heat that is delivered to the warm air outside exceeds the heat that is abstracted from the cold room). This conversion of work into heat compensates

Humpty Dumpty sat on a wall,
Humpty Dumpty had a great fall,
All the king's horses and all the king's men,
Cannot put Humpty Dumpty together again.

This is perhaps the most sensible of all the statements of the second law, for which we will allow it to pass for the moment, inasmuch as it ignores direct repair and refers at once to the most powerful of external means. It is important, however, to remember that in Humpty Dumpty's case we are concerned with structural degeneration, not with the much simpler kind of degeneration in a structureless fluid due to turbulence.

The second law of thermodynamics, of all the generalizations of physics, is certainly the most deeply seated in the common sense of all men, and one of the most humorous of children's verses refers to the man whose wondrous wisdom enabled him to circumvent it by "direct repair":

There was a man in our town
And he was wondrous wise
He jumped into a bramble bush
And scratched out both his eyes.
And when he found his eyes were out
With all his might and main
He jumped into another bush
And scratched them in again.

for the transfer of heat from the freezing room to the warm region outside.

Consider the third statement (*c*); in an ordinary steam engine heat *is* converted into work, but to accomplish this transformation a large quantity of heat must be supplied to the engine at high temperature and some of this heat (about nine-tenths of it in even the very best of steam engines) must be let down, as it were, to the low temperature of the exhaust to compensate for the conversion of the remainder into work.

50. Engines and refrigerating machines. An *engine*, or to be more specific, a *heat engine* is a machine for converting heat into

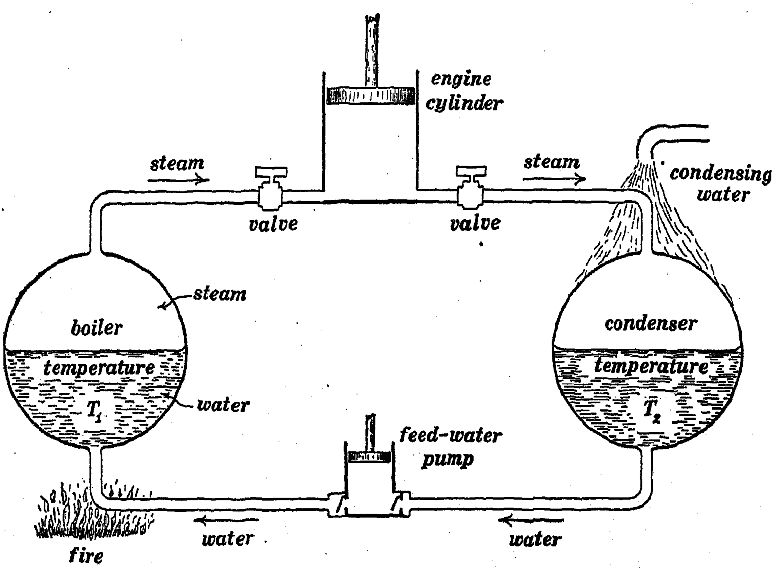


Fig. 26.

mechanical work. The essential organs of an engine are shown in Fig. 26. A certain amount of heat is supplied to the engine with the steam, a portion of this heat is transformed into work, and the remainder is delivered with the exhaust steam to the condenser.

A *refrigerating machine* is a machine for utilizing mechanical work for the extraction of heat from a cold region and its delivery

to a hot region. The essential organs of a refrigerating machine are shown in Fig. 17. The work used in driving the pump which is shown in Fig. 17 is converted into heat, and this heat, together with the heat which is extracted from the cold region, is delivered to the hot region.

The ideal or perfect engine. A most important theorem concerning the possible efficiency* of a heat engine may be established by means of an argument based on the assumption that the operation of the engine involves no irreversible or sweeping processes of any kind. The ordinary steam engine does involve such processes; the engine is subject to friction and the steam as it passes through the engine undergoes a variety of sweeping processes;† but if the engine were frictionless, if it were driven slowly, if the cylinder were prevented from cooling the steam, and if the steam were expanded sufficiently to prevent puffing, then the processes involved in the operation of the engine would be reversible. Such an ideal engine is called a *reversible engine* or a *perfect engine.*‡

During a given time the engine which is represented in Fig. 26 takes a certain quantity of heat H_1 from the boiler at temperature T_1, develops a certain amount of work W (in excess of the work required to drive the feed-water pump) and delivers the remainder of the heat H_2 to the condenser at temperature T_2.

According to the first law of thermodynamics, the work W must be equal to $(H_1 - H_2)$, H_1 and H_2 being both expressed in energy units. Therefore

$$W = H_1 - H_2 \tag{i}$$

*The efficiency of an engine is the fraction of the supplied heat which is converted into work by the engine.

†See Art. 51.

‡In order that the entire action which takes place in Fig. 26 may be reversible, the feed-water pump must be arranged to heat the water which it takes from the condenser from condenser temperature T_2 up to boiler temperature T_1. This action of the feed-water pump could be realized by arranging for it to take not only water from the condenser but also a certain amount of uncondensed steam. The compression of this mixture of steam and water under the piston of the feed-water pump would condense the steam, and the condensation would heat the feed water to boiler temperature.

For the sake of simplicity of argument the net result of the operation of the engine may be thought of as (*a*) The conversion into work the whole of the heat H_1 which is taken from the boiler at temperature T_1, and (*b*) The reconversion of the portion H_2 back into heat at temperature T_2. The regeneration* associated with process (*a*) is equal to H_1/T_1 according to equation (14), and the degeneration associated with process (*b*) is equal to H_2/T_2 according to equation (14). If the operation of the engine involves no sweeping processes, that is to say, if the engine is a reversible or perfect engine, then there can be, on the whole, no degeneration associated with the operation of the engine so that the regeneration which is associated with the process (*a*) must be balanced by the degeneration which is associated with process (*b*), or, in other words, we must have

$$\frac{H_1}{T_1} = \frac{H_2}{T_2}$$

or

$$\frac{H_1}{H_2} = \frac{T_1}{T_2} \tag{15}$$

This equation is the one upon which Lord Kelvin based his thermodynamic definition of the ratio of two temperatures. The meaning of the equation is as follows: Given a perfect engine working between boiler temperature T_1 and condenser temperature T_2; and let H_1 be the amount of heat taken by the engine from the boiler and H_2 the amount of heat delivered by the engine to the condenser in a given time. Then the ratio of H_1 to H_1 is equal to the ratio of T_1 to T_2.

Efficiency of a perfect engine. Substituting the value of H_2 from equation (i) in equation (15), and solving for W, we have

$$W = \frac{T_1 - T_2}{T_1} \cdot H_1 \tag{16}$$

*To convert an amount of work W into heat at a given temperature involves an amount of *degeneration*, and to convert the heat into work would involve the same amount of what may be called *thermodynamic regeneration*.

The fractional part $(T_1 - T_2)/T_1$ of the heat H_1 is converted into work by the engine, and this fraction is called the *efficiency* of the engine. It follows from equation (16) that *all reversible engines have the same efficiency for given values of the temperatures T_1 and T_2 whatever the nature of the working fluid may be.*

Efficiency of an imperfect engine. If the operation of the engine involves sweeping processes of any kind, the degeneration H_2/T_2 which is associated with process (*b*) above mentioned must be greater than the regeneration H_1/T_1 which is associated with process (*a*); or, in other words

$$\frac{H_1}{T_1} < \frac{H_2}{T_2}$$

or, substituting the value of H_2 from equation (i) and solving for W, we have

$$W < \frac{T_1 - T_2}{T_1} \cdot H_1$$

Therefore, comparing this inequality with equation (16), it follows that *any irreversible engine (an engine which involves sweeping processes of any kind in its operation) working between temperatures T_1 and T_2 has less efficiency than a reversible engine working between the same temperatures.*

*The refrigerating machine.** Figure 27 is a diagram for fixing in the reader's mind the various temperatures and quantities of heat and work which are involved in the operation of the refrigerating machine. The machine is representing by MM. The chamber in which evaporation takes place (the boiler) is at low temperature T_2, and the chamber in which condensation takes place (the condenser) is at high temperature T_1. The result of the operation of the machine for a given time is the abstraction of a defininite quantity of heat H_2 from the cold region, the conversion into heat of the definite amount of work W which has been used to drive the machine, and the delivery of an amount of heat H_1 to the hot region.

According to the first law of thermodynamics, the heat H_1 which is delivered to the hot region is equal to H_2+W, or

$$W = H_1 - H_2 \tag{i}$$

For the sake of argument the net result of the operation of the refrigerating machine may be thought of as (*a*) the conversion into work of all of the heat H_2 which

*See Art. 18.

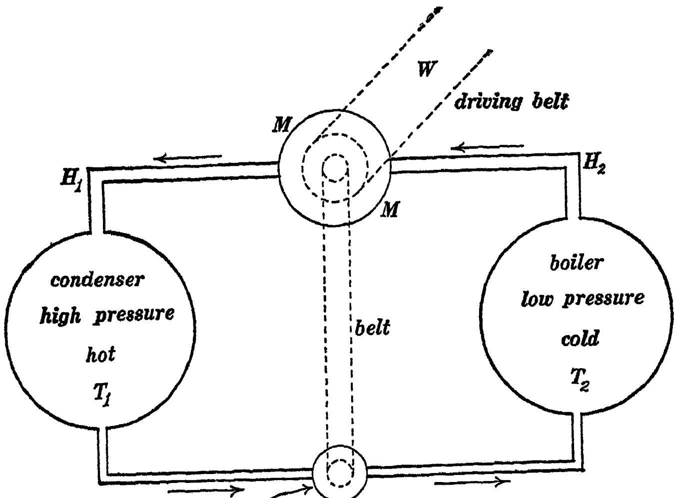


Fig. 27.

is abstracted from the cold region at temperature T_2, and (*b*) the reconversion into heat at temperature T_1 of the whole of this work (H_2) together with the work W that is expended in driving the pump. Process (*a*) involves an amount of regeneration equal to H_2/T_2, and process (*b*) involves an amount of degeneration which is equal to H_1/T_1 (where $H_1 = H_2 + W$). If all the processes involved in the operation of the refrigerating machine were reversible, its operation would not involve on the whole any degeneration, and therefore we would have

$$\frac{H_2}{T_2} = \frac{H_1}{T_1}$$

or, substituting the value of H_1 from equation (i) and solving for H_2 we have

$$H_2 = \frac{T_2}{T_1 - T_2} \cdot W \qquad (17)$$

in which H_2 is the amount of heat abstracted from the cold room by a perfect refrigerating machine during the expenditure of an amount of work W in driving the machine, T_2 being the temperature of the region from which the heat is taken and T_1 the temperature of the region to which the heat is delivered.

51. Conditions which limit the efficiency of engines in practice.* A fraction of the heat which is delivered to an engine is

*Friction between moving parts of an engine has nothing to do with the processes undergone by the steam as it passes through the engine, and therefore friction is not mentioned in the following discussion. Friction does, however, have an important influence on efficiency.

converted into work. In order that this fraction may be large, the ratio T_1/T_2 must be as large as possible, and sweeping processes must be obviated as much as possible in the operation of the engine as explained in Art. 50; T_1 being the temperature of the steam supplied to the engine and T_2 the temperature of the exhaust steam. The ratio of the initial temperature to the final temperature of the expanding steam (or gas) in the engine depends upon the ratio of the initial volume to the final volume of the steam (or gas).

In order that moderately small* cylinders may be used for the development of a given amount of power, the initial pressure of the steam or gas must be high; and in order that the final temperature may not be lower than atmospheric temperature or lower than available condenser-water temperature, the initial temperature of the steam or gas must be high.

The boiler temperatures used in ordinary steam engine practice are from 170°C. (which corresponds to about 100 pounds per square inch above atmospheric pressure) to 200°C. (which corresponds to about 200 pounds per square inch above atmospheric pressure).† The lowest feasible exhaust steam temperature is about 40°C. In the gas engine the initial temperature is the temperature of the mixture of air and gas immediately after the explosion which may be 1700°C. or higher on the "absolute" scale, and the temperature is reduced by expansion to about half this value. The mixture of gas and air in a gas engine is

*The objections to large cylinders are: (*a*) Their great cost, (*b*) the great amount of heat radiated by them, (*c*) the great amount of cylinder condensation in a large cylinder as explained later, (*d*) the great amount of piston friction and (*e*) cost of lubrication of piston.

†A boiler explosion is especially disastrous when the boiler contains a large quantity of water, and therefore the only type of boiler which is considered safe for very large pressures (and high temperatures) is the so-called *flash boiler* which is used in some steam automobiles. This boiler consists of a coil of small iron pipe; feed water is pumped into such a boiler continuously and converted almost at once into steam upon its entrance into the boiler. The temperature of the steam delivered by such a boiler is frequently as high as 250°C. which corresponds to a pressure of about 500 pounds per square inch.

always compressed before the explosion in order to enable an engine with small-sized cylinders to develop a large amount of power.

The sweeping processes which the steam in a steam engine undergoes are as follows:

(*a*) *Wire drawing.* If the pipes and passages traversed by the steam from the boiler to the engine are small, the pressure in the cylinder with open ports will be lower than boiler pressure, so that the entering steam passes from a region of high pressure into a region of low pressure. Also as the cut-off valve closes, steam will rush into the cylinder through a narrowing aperture. This effect is called wire drawing, and to provide against loss of efficiency from this cause, the pipes must be of ample size and the cut-off valve must operate very quickly.

(*b*) *Radiation.* The cooling of pipes and cylinder by the giving of heat to surrounding cooler bodies is a sweeping process, and is to be obviated as much as possible by covering pipes and cylinder with a thick coating of porous insulating material.

(*c*) *Cylinder condensation.* As a charge of steam in the cylinder expands it cools and cools the cylinder and piston, so that when steam is next admitted it heats the cylinder and piston up again and is itself cooled. This effect cannot be eliminated, but it can be largely reduced by providing separate passages for the ingress and egress of steam and by using a series of cylinders of increasing size, the smallest cylinder being arranged to take steam directly from the boiler and exhaust into the next larger cylinder which in turn exhausts into a still larger cylinder, and so on. In this way the range of temperature in each cylinder is small and the effects of cylinder condensation are greatly reduced. A steam engine in which expansion of the steam takes place in two stages (in two cylinders) is called a compound engine. A steam engine in which the expansion of the steam takes place in three stages (in three cylinders) is called a triple expansion engine.

The loss of efficiency due to cylinder condensation is greatly reduced by the use of superheated steam because the exchange of heat between the steam and the cylinder walls is very greatly

reduced when the steam does not condense. Thus S. LeRoy Brown has found that heat is imparted to a cool metal surface about forty times as fast by condensing steam as by a gas at the same temperature.

(*d*) *Effect of high piston velocity.* If the piston speed is too great, the pressure of the expanding steam becomes ineffective because the portions of the steam near the moving piston are expanded and cooled before the more remote parts of the steam are affected. This effect is negligible at the highest piston velocities which are mechanically feasible.

(*e*) *Puffing.* The steam at the end of a stroke is usually at a pressure which exceeds the pressure in the condenser and it rushes through the exhaust port as a sharp puff. This effect can be avoided by sufficiently reducing the steam pressure by expansion, but expansion should never be carried so far as to give a force on the piston (due to the steam) less than the frictional drag on piston and cross-head.

The greatest items of waste in the ordinary sense of actual loss of heat are (*a*) the incomplete combustion of the fuel, and (*b*) the carrying away of great quantities of heat in the flue gases. The economic use of fuel for the production of mechanical power requires therefore a properly designed furnace and intelligent and careful stoking to insure complete combustion, and it requires a sufficient exposure of boiler surface and frequent cleaning of the same to facilitate the flow of heat from the hot gases into the boiler.

The most pronounced sweeping process which intervenes between the completed combustion and the final exhaust of the steam from the engine is the flow of heat from the very high temperature of the fire in the furnace to the moderately low temperature of the water in the boiler, and the greatest waste in the operation of the steam engine in the sense of loss of availability of heat for conversion into work is involved in this sweeping process and it can hardly be avoided in the steam engine because of the danger involved in the generation of steam at very high pressures (and temperatures) in a large boiler.

The best gas engines convert about thirty per cent. of the heat of the fuel into mechanical work. The best steam engines convert about fifteen per cent. of the fuel into mechanical work. The ordinary run of steam engines convert only five or eight per cent. of the heat of the fuel into mechanical work.

PROBLEMS.

53. Calculate the thermodynamic degeneration in joules per degree centigrade which is represented by the conversion of mechanical energy into heat in an electric lamp which takes 50 watts and burns for one hour in a region where the temperature is 0°C. Ans. 659.34 joules per degree centigrade.

54. How much additional thermodynamic degeneration would be involved in the flow of the heat produced in the lamp in problem 53 to a region at a temperature of 30° below zero centigrade? Ans. 81.4 joules per degree centigrade.

55. Assuming that thermodynamic degeneration is a true measure of waste,* find how much waste there is in generating heat at a temperature of 1500°C. (reckoned from ice temperature) in a stove and using it in a room at a temperature of 20°C., and find how much waste is involved in turning this heat out of doors at a temperature of −10° below 0°C. Ans. 0.00285 joules per degree centigrade per joule; 0.00325 joules per degree centigrade per joule.

56. Heat is produced in a furnace under a boiler at a temperature of 1500°C. (reckoned from ice point). The boiler temperature is 180°C. (reckoned from ice point) and the temperature of the condenser of a steam engine is 70°C. (reckoned above ice point). How much waste is represented by the flow of 10,000 calories of heat from furnace into boiler due to the drop in temperature and how much waste would be represented by the flow

*Energy cannot be wasted, because, after it is converted into heat it still exists, or if heat flows from a region of high temperature to a region of low temperature, the heat is not wasted because it remains unchanged in amount. The only waste represented in ordinary physical processes is that which is measured by thermodynamic degeneration.

of 10,000 calories from boiler to condenser due to drop in temperature? Ans. 16.44 calories per degree centigrade; 7.07 calories per degree centigrade.

57. A perfect engine taking steam at 160°C. and exhausting at 70°C. would have what efficiency? A good steam engine working between these temperatures has efficiency equal to 75 % of the efficiency of a perfect engine. What is the actual efficiency of the steam engine? Ans. 20.8 per cent. 15.6 per cent.

58. The mixture of air and gas in a gas engine reaches a temperature of 1100°C. (reckoned above ice point) when it is exploded, and the temperature of this mixture is reduced to, say, 600°C. (reckoned above ice point) by expansion in the engine. What would be the efficiency of a perfect engine working between these temperatures? Ans. 36.4 per cent.

59. An ammonia refrigerating machine is required to take heat from a room at −10°C. and the pipes in which the ammonia vapor is condensed are kept at temperature of 35°C. by being exposed to the air. Assuming the refrigerating machine to be a perfect engine, calculate the work in kilowatt-hours required to freeze one kilogram of ice from ice water. Assuming the refrigerating machine to take 1½ times as much work as a perfect engine, calculate the work required to freeze one kilogram of ice from ice water. Express the result in kilowatt-hours and in horse-power-hours. Ans. 0.016 kilowatt-hours; 0.024 kilo-watt hours; 0.0214 horse-power-hours; 0.0322 horse-power-hours.

CHAPTER VI.

FURTHER DEVELOPMENTS OF THERMODYNAMICS.

[Throughout this chapter c.g.s. units are employed unless it is explicitly stated to the contrary; that is, pressures are expressed in dynes per square centimeter, volumes in cubic centimeters, masses in grams, and work and heat are both expressed in ergs.]

52. Specification of physical state. Watt's diagram. The physical state of a given fluid,* the physical state of water, for example, is completely fixed when its pressure and volume-per-gram are given.† It is a great help to represent the physical state of a fluid by a point A, Fig. 28, of which the abscissa represents volume-per-gram and of which the ordinate represents pressure. *In most of the following diagrams, however, abscissas represent total volumes of a given amount of the fluid and ordinates represent pressures.* Such a diagram is called a *Watt's diagram.*

When a fluid expands it does work if allowed to push against a piston; the work done by the fluid during an increment of volume Δv is equal to $p \cdot \Delta v$, and it is represented by the shaded area in Fig. 28.‡ To show that the work done by an expanding fluid is equal to $p \cdot \Delta v$, imagine the fluid to expand against a piston of area a. The force with which the fluid pushes on the piston is equal to pa. The product of this force times a small distance Δx moved by the piston is the work done by the expanding fluid. Therefore the work done by the expanding fluid is equal to $pa \cdot \Delta x$; but $a \cdot \Delta x$ is the increment of volume of the fluid and therefore

$$pa \cdot \Delta x = p \cdot \Delta v.$$

Process curve. Work done by a fluid during a process. A substance which passes slowly from one physical state to another because of changing pressure or changing volume, or both, undergoes a reversible process. The points in Watt's diagram which represent the successive states passed through by a fluid during a reversible process form a continuous§ curve called a process curve. The work done by the fluid

*The thermodynamics of solids is very complicated. The object of this treatise is to bring out some of the most important facts of thermodynamics in the simplest possible way and therefore the details of this discussion are limited chiefly to fluids.

†The physical state of a fluid is also fixed when its pressure and temperature are given, or when its temperature and volume-per-gram are given. It is usually most convenient to specify physical state in terms of *pressure* and *volume-per-gram* because of the simplicity of the expression for work done by an expanding fluid in terms of pressure and volume.

‡On the assumption that abscissas in Fig. 28 represent total volumes of a given amount of fluid.

§Reversible processes, only, can be represented in Watt's diagram. During an irreversible process (a sweeping process) the fluid has no definite pressure and

during the process AB is represented by the shaded area in Fig. 29; this work may be done on a piston or it may be done in pushing back surrounding air, when, for example, water is converted into steam in an open vessel. During an increase

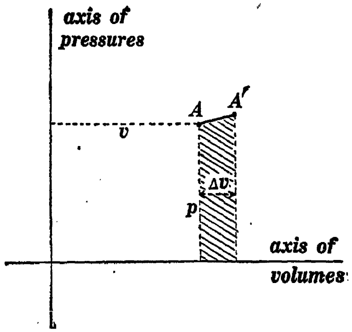


Fig. 28.

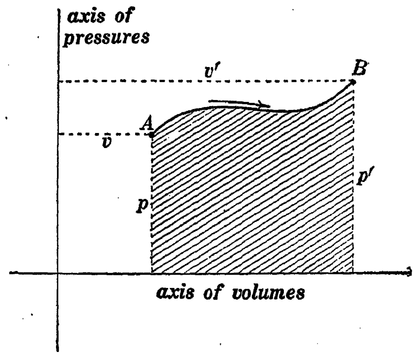


Fig. 29.

of volume work is done *by* the fluid, and during a decrease of volume work is done *on* the fluid.

Cyclic process. A process is said to be cyclic when the substance comes back to its initial state at the end of the process. A curve in Watt's diagram which represents a cyclic process is a closed curve. An important example of a cyclic process is discussed in the next article.

53. Carnot's cycle. A given portion of water goes through a cyclic process in a steam engine in which the condensed water is pumped back into the boiler. If each stage of this cyclic process is performed slowly without the giving of heat to surrounding bodies by the steam or the receiving of heat from surrounding bodies by the steam, and if the feed water pump is arranged to heat the water from condenser temperature to boiler temperature, as explained in Art. 50, then the cyclic process through which the steam passes is reversible and constitutes what is called a *Carnot cycle.**

The water† goes through the following processes as it passes through the Carnot cycle:

(*a*) The water is expanded at constant temperature and pressure as it evaporates in the boiler due to passage of some steam into the engine cylinder;

(*b*) The supply of steam is then shut off from the engine and the steam in the cylinder is expanded, causing its temperature and pressure both to fall to the temperature and pressure which prevail in the condenser;

(*c*) The exhaust ports are then opened and the cylinder-full of steam (at con-

no definite temperature. If a portion of the process which carries a fluid from state A to state B, Fig. 29, is a sweeping process, a gap exists in the process curve AB.

*So called from Sadi Carnot, the first to apply what is now called the second law of thermodynamics to the theory of the steam engine.

†This term is here intended to include both the liquid form and the vapor form.

denser pressure) is forced into the condenser by the returning piston, that is to say, the steam in condenser and cylinder is reduced in volume by the returning piston and nearly all condensed to water; and

(*d*) The water from the condenser together with a certain residue of uncondensed exhaust steam is received by the feed-water pump and compressed. This compression causes the residue of steam to condense, thus raising the temperature of the feėd-water up to boiler temperature as it leaves the feed-water pump.

These four processes may be more briefly described as follows:

(*a*) Expansion at constant temperature;

(*b*) Further expansion without the giving of heat to the working fluid,

(*c*) Compression at constant temperature, and

(*d*) Further compression without the taking of heat from the working fluid.

During process (*a*) a certain amount of heat H_1 is given to the working fluid to keep its temperature constant, and during process (*c*) a certain amount of heat H_2 is taken from the working fluid to keep its temperature constant. If the four process (*a*), (*b*), (*c*) and (*d*) are performed slowly and without waste of heat to surrounding bodies, that is if the four processes are performed reversibly, then we have

$$\frac{H_1}{H_2}=\frac{T_1}{T_2} \tag{i}$$

as explained in Art. 50. A more convenient form of this equation for subsequent use is

$$\frac{H_1}{T_1}=\frac{H_2}{T_2} \tag{ii}$$

or, considering that H_1 is heat *given to* the working fluid and that H_2 is heat *taken from* the working fluid, it is permissible to think of H_1 as being positive and H_2 as being negative, so that we may write

$$\frac{H_1}{T_1}+\frac{H_2}{T_2}=0 \tag{18a}$$

The Carnot cycle as performed by a portion of water* in the steam engine is represented in Fig. 30. The line AB represents process (*a*), BC represents process (*b*), CD represents process (*c*) and DE represents process (*d*). The expansion at constant temperature which is represented by AB takes place without change of pressure because this expansion is accompanied by the evaporation of water in the boiler; and the compression at constant temperature which is represented by the line CD takes place without change of pressure because of the condensation of the steam in the condenser. The work done by the steam on the piston during process (*a*) is represented by the area $ABA'B'$. The work done by the steam on the piston during process (*b*) is represented by the area $BCB'C'$. The work done by the piston on the steam, that is to say, the work given back to the steam during the process (*c*) is represented by the area $CDC'D'$, and the work given back to the steam during process (*d*) is represented by the area $ADA'D'$. Therefore the enclosed area $ABCD$ represents the work obtained during the cycle, and this work is equal to H_1-H_2 as explained in Art. 50.

*This term is here intended to include both the liquid form and the vapor form.

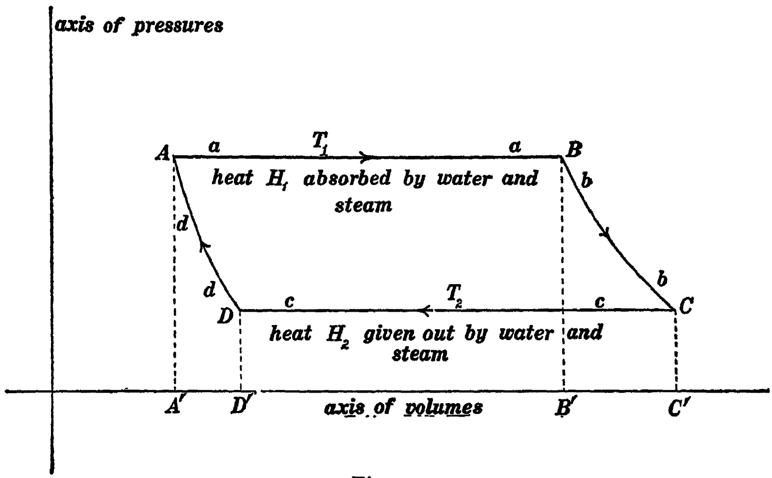


Fig. 30.

A gas may be made to perform a Carnot cycle. In this case process (*a*) (expansion at constant temperature) takes place with falling off of pressure because the pressure of a gas is inversely proportional to the volume when the temperature is constant; and process (*b*) (compression at constant temperature) takes place with rise of pressure for the same reason. Fig. 31 represents a Carnot cycle as performed by a gas.

A curve showing the relationship between volume and pressure of a gas at constant temperature is called an *isothermal curve*; the curves *a* and *c* in Fig. 31 are

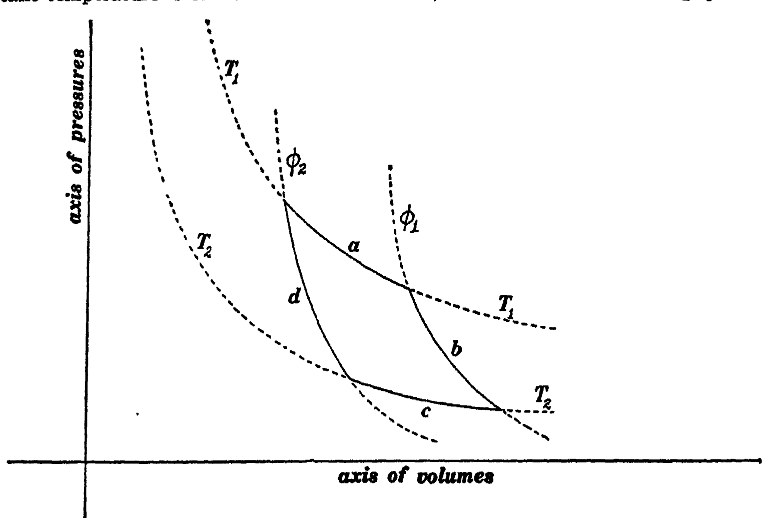


Fig. 31.

isothermal curves corresponding to temperatures T_1 and T_2. A curve which represents the relation between volume and pressure of a gas when it expands without receiving or giving off heat is called an *adiabatic curve* (or *isentropic curve*); curves b and d in Fig. 31 are portions of the adiabatic curves ϕ_1 and ϕ_2.

54. Clausius's theorem. Entropy of a substance. Consider a fluid which, starting from any given pressure p and volume v as shown in Fig. 32, is subjected

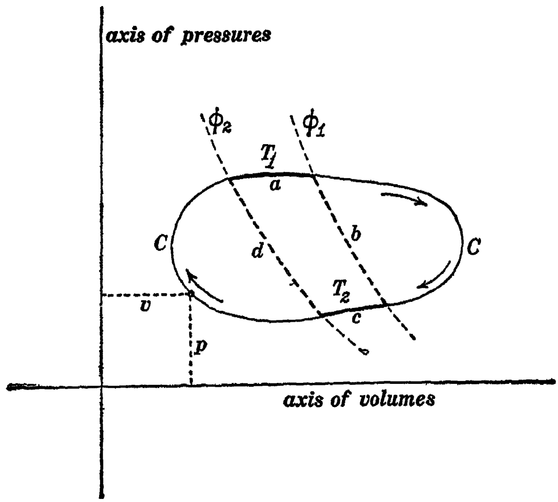


Fig. 32.

to any combination whatever of slow heating (or cooling) and expanding (or contracting) so that its changing pressure and volume may be represented by the successive ordinates of the closed curve CC. The process undergone by the fluid is reversible because it is performed very slowly, and it is a cyclic process because the substance comes back to its initial state. Consider two adiabatic curves ϕ_1 and ϕ_2 *very close together*, and consider the cyclic process which is represented by the heavy lines a, b, c, and d. This process constitutes sensibly a Carnot cycle because the portions a and c of the curve are so short that any slight change of temperature which takes place while the fluid is passing over these portions is negligible. Let ΔH_1 be the heat given to the fluid while it is passing over the short curve a, and let T_1 be the temperature of the fluid during this process; let ΔH_2 be the heat taken from the fluid while the fluid is passing over the short curve c, and let T_2 be the temperature of the fluid during this process. Then, according to equation (18a), we have

$$\frac{\Delta H_1}{T_1} + \frac{\Delta H_2}{T_2} = 0 \tag{1}$$

in which ΔH_1, being heat delivered to the fluid, is considered as positive, and ΔH_2, being heat taken from the fluid, is considered negative.

The whole of the given cyclic process may be broken up into pairs of corre-

sponding parts like *a* and *c*, Fig. 32, so that the entire heat taken in by the fluid and given out by the fluid during the different parts of the given cyclic process may be arranged in pairs like ΔH_1 and ΔH_2, and each pair of heat-parts satisfies an equation like equation (i). Therefore *the sum of all quotients obtained by dividing the heat delivered to the fluid (heat taken from the fluid being considered as negative) at each step of any reversible cyclic process by the absolute temperature of the fluid at the step is equal to zero.* That is

$$\Sigma \frac{\Delta H}{T} = 0 \tag{18b}$$

Consider two states of thermal equilibrium of a fluid A and B, Fig. 33, p and v being the pressure and volume of the fluid in state A, and p' and v' being the pressure and volume of the fluid in state B. Let the curves *aa* and *bb* Fig. 33 represent

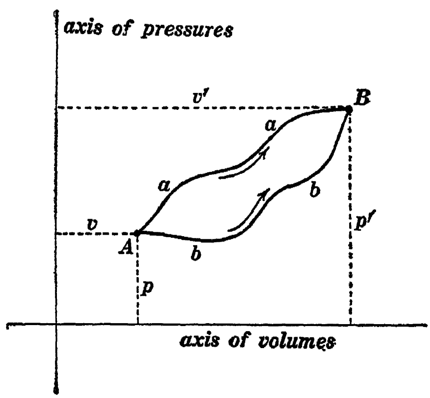


Fig. 33.

any two different reversible processes leading from A to B. Then process *bb* reversed and process *aa* together constitute a cyclic process starting from A and returning to A. Therefore the sum $\Sigma\Delta H/T$ is equal to zero when it is extended over process *aa* and over process *bb reversed.* Therefore the sum $\Sigma\Delta H/T$ extended over process *aa* is equal and opposite to the sum $\Sigma\Delta H/T$ extended over process *bb reversed*, but the sum $\Sigma\Delta H/T$ extended over process *bb reversed* is equal and opposite to $\Sigma\Delta H/T$ extended over process *bb not reversed*, because *heat given* to a fluid (or taken from a fluid) during any step of a given process becomes *heat taken from* the fluid (or heat given to the fluid) if the process is reversed; that is to say, ΔH for each step of a reversed process is opposite in sign to the corresponding value of ΔH for the process not reversed.

The above argument leads at once to the proposition that $\Sigma\Delta H/T$ *has the same value for any two, and therefore for all reversible processes which lead from one given state of thermal equilibrium A to another given state of thermal equilibrium B.* This theorem is due to Clausius.

Consider a portion of heat ΔH given to (or taken from) a substance during a small part of a reversible process and imagine this heat to be produced by the degen-

eration of work into heat (or if ΔH is heat taken from the substance imagine it to be regenerated into work). Then $\Delta H/T$ would be the thermodynamic degeneration (or regeneration) associated with the small part of the reversible process, due, of course, not to turbulence in the substance which is undergoing the reversible process but to the actions outside of the substance which are involved in the assumed production of ΔH by degeneration of work into heat (or the actions which are involved in the regeneration of ΔH into work). Therefore the sum $\Sigma\Delta H/T$ for any reversible process like *aa* or *bb* Fig. 33 which carries a fluid from a given state A to a given state B is *a measure of the thermodynamic degeneration which is associated with the* **difference in state** *between* A *and* B. This thermodynamic degeneration which is associated with a difference in state is called the *entropy difference* between the two states. Thus, if state B can be reached from A by a sweeping process, then $\Sigma\Delta H/T$ has a positive value for any reversible process leading from A to B and this value of $\Sigma\Delta H/T$ is called the *entropy* of the state B referred to the state A.

Let ϕ be the entropy difference between two states of a substance, then according to the above definition of entropy difference we have

$$\phi=\Sigma\frac{\Delta H}{T} \tag{19}$$

where $\Delta H/T$ refers to an element of a reversible process leading from the one state of the substance to the other, ΔH being the heat delivered to the substance during this element or step and T being the temperature of the substance during the step. If the two states are very near together then equation (19) may be written in the form

$$\Delta\phi=\Delta H/T \tag{20}$$

in which $\Delta\phi$ is the entropy difference between the two adjacent states of a substance, ΔH is the heat which must be delivered to the substance to change it from one state to the other, and T is the absolute temperature of the substance.*

Entropy differences, only, have a physical significance. A certain state of fluid may be arbitrarily chosen as a zero state or reference state, and the value of the entropy of the substance in any other given state may be defined as the value of $\Sigma\Delta H/T$ extended over any reversible process leading from the zero or reference state to the given state. This is equivalent to assigning arbitrarily the value zero to the entropy of the substance in the zero state.

Example. A gas is slowly expanded under a piston at constant temperature T, and a quantity of heat H is given to the gas to keep its temperature constant. Then, since T is constant during the process, H/T is the increase of entropy of the gas during the expansion. That is to say there is an entropy difference of H/T between the initial and final states of the gas. The same change of state may be brought about by allowing the gas to expand freely (through an orifice), and the thermodynamic degeneration involved in this sweeping process is measured by the entropy difference between the initial and final states of the gas.

The integral $\Sigma\cdot\Delta H/T$ cannot be applied to a sweeping process because a sub-

*Of course the temperature in one state may differ from the temperature in the other state by an infinitesimal amount.

stance which is undergoing a sweeping process is not at any time in thermal equilibrium during the process and the substance therefore has no temperature during the process. The thermodynamic degeneration which is involved in a simple sweep is the entropy difference between the initial and final states of the substance, and it can be evaluated only by calculating the value of the integral $\Sigma \cdot \Delta H / T$ during a *reversible* process which leads from the same initial to the same final state of the substance.

. *The entropy of a substance in a given state is proportional to the mass of the substance.* This is evident when we consider that the value of ΔH for every step of a reversible process is doubled if the mass of the fluid is doubled. Entropy is expressed in units of heat per degree of temperature as explained in Art. 48.

55. Specific heats of a gas.* The number of thermal units (ergs) required to raise the temperature of one gram of a gas one degree is called the *specific heat* of the gas. If the volume of the gas does not change during the rise of temperature, then, inasmuch as no external work is done by the gas, all of the heat applied goes to increase the internal energy of the gas. If, however, the gas be allowed to expand as the temperature rises, to such an extent, for example, as to keep the pressure constant, then the heat which is supplied to the gas to raise its temperature must not only increase the internal energy of the gas but must also make up for the external work done by the gas as it expands. The specific heat of a gas has therefore two important values, namely, the speicfic heat at constant volume C_v, and the specific heat at constant pressure C_p, of which the latter has the larger value.

Relation between C_v and C_p. To raise the temperature of M grams of a gas ΔT degrees requires $C_vM \cdot \Delta T$ units of heat (ergs) if the volume is kept constant. If the gas is then allowed to expand by an amount Δv sufficient to bring the pressure back to its initial value p, an amount of work equal to $p \cdot \Delta v$ is done by the expanding gas and an amount of heat equal to $p \cdot \Delta v$ must be given to the gas to keep up its temperature, for, according to Joule and Thomson's principle, the only appreciable cause of change of temperature by expansion of a gas is the loss of energy of the gas by the doing of external work. Therefore a total amount of heat equal to $C_vM \cdot \Delta T + p \cdot \Delta v$ is required to raise the temperature of the gas by the amount ΔT when the gas is allowed to expand sufficiently to keep its pressure constant; but according to the definition of C_p the amount of heat required for this change of temperature at constant pressure is equal to $C_pM \cdot \Delta T$. Therefore

$$C_pM \cdot \Delta T = C_vM \cdot \Delta T + p \cdot \Delta v \tag{i}$$

*Not only does a gas have two specific heats according as it is kept at constant volume or at constant pressure during the increase of temperature, but a gas has two important values for its bulk modulus (see *Mechanics*, Art. 99). If the temperature of the gas is kept constant during compression by the extraction of heat from the gas, then the rise of pressure due to a given decrease of volume is less than if no heat is abstracted from the gas during compression. In the one case we have what is called the *isothermic bulk modulus*, and in the other case we have what is called the *isentropic* or *adiabatic bulk modulus*.

All substances have two values of specific heat, namely, a specific heat at constant volume and a specific heat at constant pressure, and every substance has isothermic and isentropic moduli of elasticity. It is only in the case of gases, however, that these differences are large enough to be appreciable.

Proposition. The constant R in equation (4*b*) of Art. 6 for a given gas is equal to the difference between the specific heat of the gas at constant pressure (C_p) and the specific heat of the gas at constant volume (C_v); that is

$$C_p - C_v = R \tag{21}$$

Proof. Imagine M grams of the gas to have been increased in temperature by the amount ΔT at constant pressure, Δv being the increment of volume. Then we have equation (i) above. The increment of volume necessary to keep the pressure constant when the temperature increases is found by differentiating equation (4*b*) remembering that T and v, only, vary. This gives

$$p \cdot \Delta v = MR \cdot \Delta T \tag{ii}$$

Substituting this value of $p \cdot \Delta v$ in equation (i), we have equation (21).

56. Isothermic expansion and compression. When the temperature of a gas is kept at a constant value by supplying heat to the gas as the gas expands, or by

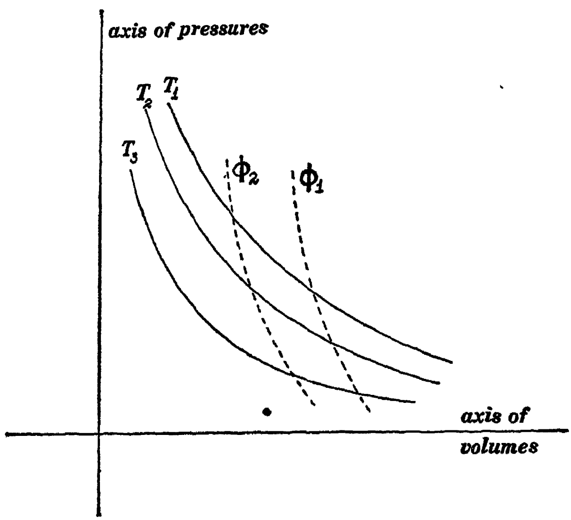


Fig. 34.

abstracting heat from the gas as the gas is compressed, the expansion or compression is said to be *isothermic*. Equation (4*b*) gives the relation between p and v of a perfect gas during isothermic expansion or compression, T being then a constant as well as M and R.

The full-line curves in Fig. 34 show graphically the isothermic relations between p and v for air for several different temperatures. The ordinates of these curves represent pressures and the abscissas represent volumes. The curves are sometimes called the *isothermal lines* of the gas; they are equilateral hyperbolas.

57. Adiabatic or isentropic expansion and compression. When a gas is expanded or compressed and not allowed to receive heat from, or to give heat to sur-

rounding bodies, the expansion or compression is called *adiabatic* or *isentropic*.*

The relation between p and v of a perfect gas during isentropic expansion or compression is:

$$pv^k = \text{a constant} \tag{22}$$

in which

$$k = \frac{C_v}{C_p} \tag{23}$$

The dotted curves in Fig. 34 show the isentropic relation between p and v for air (for two different values of the entropy ϕ_1 and ϕ_2). The value of k for air is 1.41.

Proof of equation (22). Consider a very slight isentropic expansion of a gas, and let Δv, Δp and ΔT be the actual changes of volume, pressure and temperature; Δv being an increment, while Δp and ΔT are both decrements and therefore both negative.

The work $p \cdot \Delta v$ *done by* the gas is all made up by the decrement $MC_v \cdot \Delta T$ of the internal energy of the gas;† therefore, remembering that $MC_v \cdot \Delta T$ is negative while $p \cdot \Delta v$ is positive we have:

$$p \cdot \Delta v = -MC_v \cdot \Delta T \tag{i}$$

From the equation $pv = MRT$ (4*b*) we have:

$$T = \frac{1}{MR} \cdot pv \tag{ii}$$

whence

$$\Delta T = \frac{1}{MR}(p \cdot \Delta v + v \cdot \Delta p) \tag{iii}$$

Substituting this value of ΔT in equation (i), remembering that $R = C_p - C_v$ according to equation (21), and that $k = C_p / C_v$ (23), we have:

$$\frac{\Delta p}{p} + k\frac{\Delta v}{v} = 0 \tag{iv}$$

whence by integrating

$$\text{Log } p + k \text{ Log } v = \text{a constant}$$

or

$$\text{Log } (pv^k) = \text{a constant}$$

or

$$pv^k = \text{a constant}$$

58. Proposition. *Temperature ratios as measured by a gas thermometer conform to temperature ratios as defined by equation* (15) *of Art.* 50 *if the gas which is used in the gas thermometer conforms to Boyle's Law, and if it is a gas of which the temperature neither rises nor falls during free expansion.* To establish this proposition, it is necessary to consider very carefully what assumptions and what experimental facts lie at the basis of the discussion in Arts. 55, 56 and 57.

(1) Some method of measuring temperature is necessary as a basis for the ex-

*The entropy of a gas is constant during this kind of expansion.

†This is equivalent to saying that the only cause of cooling of the expanded gas is its loss of energy by the doing of external work $p \cdot \Delta v$. See discussion of equation (21) in Art. 55.

perimental determination of specific heats. In fact all accurate work in specific heats has been based upon temperatures as measured by the gas thermometer. The use of equation (4*b*) in Arts. 55, 56, and 57 means, in the first place, that the gas under discussion is assumed to conform to Boyle's law and it means, in the second place, that temperatures as used in the discussion are supposed to be measured by a gas thermometer.

(2) The assumption is made in Art. 55 that the gas under discussion is one of which the temperature neither rises nor falls during free expansion and this assumption again appears in Art. 57.

(3) Regnault's experiments on the specific heats of gases show that the specific heat at constant pressure of any ordinary gas does not change sensibly with the temperature or pressure of the gas within a moderate range of temperatures and pressures. Therefore C_p may be treated as a constant. Therefore the specific

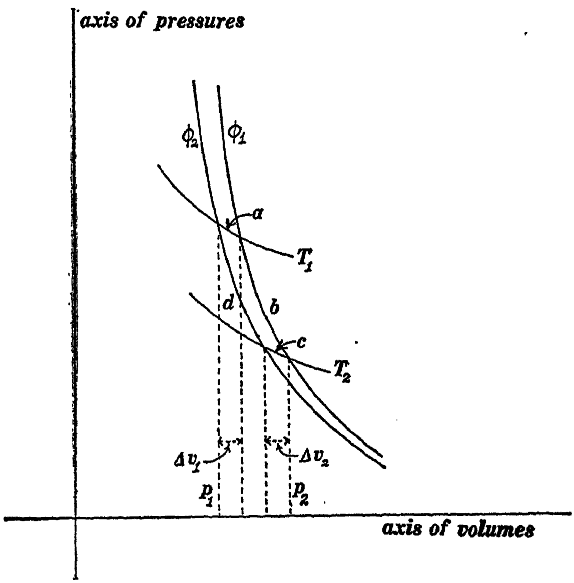


Fig. 35.

heat of any ordinary gas at constant volume, C_v, is sensibly constant for ordinary temperatures and pressures according to equation (21). In fact, the integration of equation (iv) in Art. 57 assumes the constancy of k $(= C_p/C_v)$.

To establish the above proposition consider a Carnot cycle *a b c* and *d*, between two very closely adjacent adiabatic curves ϕ_1 and ϕ_2 and between any two isothermal curves T_1 and T_2 as shown in Fig. 35. The proposition is then established if we can show that

$$\frac{\Delta H_1}{\Delta H_2} = \frac{T_1}{T_2} \qquad \text{(i)}$$

where ΔH_1 is the heat taken in by the gas during process *a*, and ΔH_2 is the heat given out by the gas during process *c* in Fig. 35.

The gas being one of which the temperature neither rises nor falls during free expansion, ΔH_1 is equal to the work done by the gas during process *a*, and ΔH_2 is equal to the work done on the gas during process *c*, and inasmuch as the two adiabatic curves ϕ_1 and ϕ_2 are very close together, this work is in each case equal to the change of volume (Δv_1 or Δv_2) multiplied by the prevailing pressure (p_1 or p_2). It remains therefore to find an expression for the increment of volume of a gas along any isothermal curve T between the two adiabatic curves ϕ_1 and ϕ_2.

The equation of one of the isothermal lines in Fig. 35 is

$$pv = MRT \tag{ii}$$

according to equation (4*b*), and the equation to the adiabatic lines is

$$pv^k = C \tag{iii}$$

according to equation (22); and to go from one adiabatic curve to another is to assign a definite increment ΔC to the quantity C. Differentiating equation (iii) we have

$$\Delta C = kpv^{k-1} \cdot \Delta v + v^k \cdot \Delta p \tag{iv}$$

and in order that the two variations Δv and Δp may correspond to a movement along an isothermal line, they must be variations obtained from the differentiation of equation (ii) on the assumption that T is constant. Therefore, differentiating equation (ii), we have

$$\Delta p = -\frac{MRT}{v^2} \cdot \Delta v$$

or, using pv for MRT, we have

$$\Delta p = -\frac{p}{v} \cdot \Delta v \tag{v}$$

Substituting this value of Δp in equation (iv) and solving for Δv, we have

$$\Delta v = \frac{\Delta C}{(k-1)pv^{k-1}} \tag{vi}$$

Therefore the work $p \cdot \Delta v$ done on the gas while it is expanding along an isothermal curve from one adiabatic line to the other adiabatic line in Fig. 35 is

$$\Delta H = \frac{\Delta C}{(k-1)v^{k-1}} \tag{vii}$$

in which ΔH is written for the work done by the gas because the work done by the gas is equal to the heat which is imparted to the gas to keep its temperature constant. This expression for ΔH may be written

$$\Delta H = pv \cdot \frac{\Delta C}{(k-1)pv^k}$$

which may be written

$$\Delta H = MRT \cdot \frac{\Delta C}{(k-1)pv^k} \tag{viii}$$

inasmuch as pv is equal to MRT. With respect to this equation (viii) it is to be noted that the numerator has a definite value for a movement from one to another adiabatic curve, and that the denominator has the same value everywhere along

any adiabatic curve. Therefore the fraction $\Delta C/[(k-1)\,pv^k]$ has the same value during processes a and b in Fig. 35. Writing T_1 for T in equation (viii) we have an expression for ΔH_1 and writing T_2 for T in equation (viii) we have an expression for ΔH_2, and by combining these two results we find at once that $\Delta H_1/\Delta H_2$ equals T_1/T_2.

59. Rise in temperature of a gas during isentropic compression. We may eliminate p (or v) from equation (22) by substituting the value of p (or v) from equation (4b). In this way we find:

$$Tv^{k-1} = \text{a constant} \tag{24}$$

or

$$Tp^{(1-k)/k} = \text{a constant} \tag{25}$$

during isentropic expansion or compression.

Examples of isentropic expansion and compression. The expansion and compression of the air in sound waves is isentropic, because the expansion and compression take place so quickly that the expanded or compressed portion of the air does not have time to give off or to receive any appreciable amount of heat.

When a large mass of air moves upwards, as, for example, when a wind blows up a mountain slope, the pressure of the mass of air falls off with increasing altitude and the temperature is reduced. The expansion is isentropic, inasmuch as the mass of air is so large that it cannot, during the brief time of the ascent, receive or give off an appreciable amount of heat. Suppose that the rising mass of air was initially at temperature T_1 and pressure p_1, and that its pressure has fallen to p_2. The temperature T_2 corresponding to p_2 may be found from equation (25), which gives

$$T_1p_1^{(1-k)/k} = T_2p_2^{(1-k)/k} \tag{26}$$

A very interesting phenomenon due to the cooling of a rising column of air by isentropic expansion is the formation of the beautiful cumulous clouds on a quiet summer day. The warm moist air near the ground starts streaming upwards through the superimposed cold air. This upward stream once started draws like a chimney and the rising column develops until it becomes very large and very high. At a very sharply defined altitude the pressure reaches a value for which the temperature [according to equation (26)] of the rising air is reduced to the dew point. The further cooling which is produced as the air passes above this level, causes the condensation of water vapor and the formation of mist or cloud. The strikingly flat bottom of a cumulous cloud is at that altitude where the rising air reaches the temperature of the dew point.

The condensation of water vapor as rain is due very largely to the isentropic cooling of the great rising column of air, sometimes hundreds of miles in diameter, at the center of what is technically called a cyclone (See Art. 10).

When a gas is quickly compressed under a piston in a cylinder, the compression is isentropic. If the initial volume is v_1, initial temperature T_1, and final volume v_2, we may find the temperature T_2 corresponding to v_2 from equation (24), which gives

$$T_1v_1^{k-1} = T_2v_2^{k-1} \tag{27}$$

60. Method of Clement and Desormes for the experimental determination of k. A vessel filled with the gas at pressure p_1 and temperature T_1 is opened to the air,

allowing the pressure to drop quickly to atmospheric pressure p_2, the temperature falling at the same time to T_2. Equation (26) then gives

$$T_1 p_1^{(1-k)/k} = T_2 p_2^{(1-k)/k} \tag{26 bis}$$

The vessel is then quickly closed and allowed to stand until it has reached its former temperature T_1, when the pressure is p_3, so that from Gay Lussac's law we have

$$\frac{T_2}{p_2} = \frac{T_1}{p_3} \tag{i}$$

Substituting the value of T_2 from (i) in equation (26) and reducing, we have

$$\left(\frac{p_1}{p_2}\right)^{\frac{1-k}{k}} = \frac{p_2}{p_3} \tag{ii}$$

from which k may be calculated, p_1, p_2, and p_3 having been observed. The value of k for air is 1.41.

Remark. The numerical value of the constant R equation (4*b*), is easily calculated from the observed value of the density M/v of the gas at a known temperature T and pressure p. Therefore the value of $(C_p - C_v)$ is known for the gas from equation (21). The difference $(C_p - C_v)$ being thus known and the ratio C_p/C_v being determined by Clement and Desormes' method, one may easily calculate the values C_p and C_v.

61. Entropy of one gram of gas expressed in terms of its pressure and volume. An expression for the entropy of one gram of a gas in terms of its pressure and volume may be derived with the help of the relation

$$\Delta\phi = \frac{\Delta H}{T} \tag{i}$$

in which $\Delta\phi$ is the increase of entropy of the substance when an amount of heat ΔH is given to the substance at temperature T. The meaning of this equation may be understood by a careful study of Art. 54.

The amount of gas under consideration being one gram, equation (4*b*) becomes

$$pv = RT \tag{ii}$$

so that the change of temperature ΔT due to a given change of volume Δv at constant pressure is found by differentiating equation (ii) with respect to v, giving

$$\Delta T = \frac{p \cdot \Delta v}{R} \tag{iii}$$

The change of temperature ΔT corresponding to a change of pressure Δp at constant volume is found by differentiating equation (ii) with respect to p, giving:

$$\Delta T = \frac{v \cdot \Delta p}{R} \tag{iv}$$

Change of entropy due to change of volume at constant pressure. Equation (iii) gives the change of temperature produced by change of volume Δv at constant pressure, and an amount of heat, $\Delta H = C_p \ \Delta T$, must be given to the gas to keep

the pressure constant during this change of volume. Therefore using the value of ΔT from equation (iii) we have

$$\Delta H = \frac{C_p \cdot p\Delta v}{R}$$

and substituting this value of ΔH in equation (i) we have

$$\Delta\phi = C_p \cdot \frac{\Delta v}{v} \tag{v}$$

in which pv has been used for RT.

Change of entropy due to change of pressure at constant volume. Equation (iv) gives the change of temperature due to change of pressure Δp at constant volume and an amount of heat, $\Delta H = C \cdot \Delta T$, must be given to the gas to produce the increase of pressure at constant volume. Therefore using the value of ΔT from equation (iv) we have

$$\Delta H = \frac{C_v \cdot \Delta p}{R}$$

and substituting this value of ΔH in equation (i) we have

$$\Delta\phi = C_v \cdot \frac{\Delta p}{p} \tag{vi}$$

in which pv has been used for RT.

Complete differential of entropy. Equation (v) expresses the differential of entropy with respect to volume, and equation (vi) expresses the differential of entropy with respect to pressure; therefore the complete differential of entropy is given by the equation

$$\Delta\phi = C_p \frac{\Delta v}{v} + C_v \cdot \frac{\Delta p}{p} \tag{vii}$$

which by integration gives

$$\phi = C_p \log_e v + C_v \log_e p \tag{28}$$

from which the constant of integration is omitted inasmuch as differences of entropy, only, have physical significance as explained in Arts. 53 and 54.

62. Application of equation (15) to the vaporization of water. Consider one gram of water in the form of saturated steam at temperature T, the pressure and volume of the steam being p and V_s, respectively, as represented by the co-ordinates of the point a in Fig. 36. Imagine this one gram of water to be carried through a Carnot cycle as follows:

(1) Compress the steam at constant temperature T until it is nearly all condensed into water at the point b in Fig. 36. During this condensation an amount of heat H_2 is abstracted from the steam.

(2) Compress the condensed water at point b (together with an infinitesimal residue of uncondensed steam), without abstracting heat from it, until its temperature and pressure rise to $T+\Delta T$ and $p+\Delta p$, respectively, as represented by the co-ordinates of the point c in Fig. 36.

(3) Allow the water to be converted into steam by increasing its volume, the temperature being kept at $T+\Delta T$ by giving an amount of heat H_1 to the water. This process of vaporization is represented by the line cd in Fig. 36.

(4) Allow the steam (together with an infinitesimal residue of unvaporized

water) to expand, without receiving heat, until its temperature and pressure fall to the initial temperature and pressure. This process is represented by the line *da* in Fig. 36.

The heat H_2 taken from the water in part 1 of the cycle is equal to the latent heat of vaporization L of water at the given temperature and pressure because it is the amount of heat given off by one gram of steam in being condensed to water at a fixed temperature; and the heat H_1 given to the water during part 3 of the

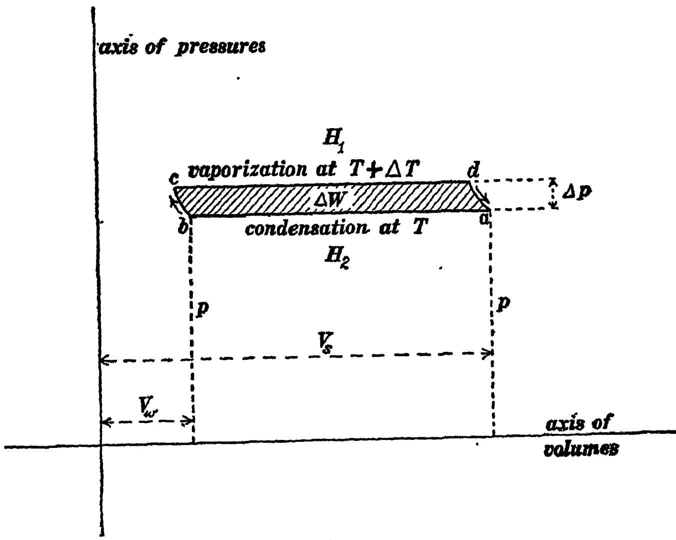


Fig. 36.

cycle exceeds H_2 by an amount which is equal to the work ΔW gained during the cycle; this work is represented by the shaded area in Fig. 36 and it is equal to $(V_s - V_w) \cdot \Delta p$, where V_w is the volume of one gram of liquid water at temperature T. Therefore, using L instead of H_2, equation (15) gives

$$\frac{L}{L + (V_s - V_w) \cdot \Delta p} = \frac{T}{T + \Delta T}$$

whence

$$\frac{\Delta p}{\Delta T} = \frac{L}{T(V_s - V_w)} \tag{29}$$

This equation expresses the steepness $(\Delta p/\Delta T)$ of the vapor-pressure curve in terms of the latent heat of vaporization L of the liquid, the absolute temperature T of the boiling point, and the increase of volume of one gram of the liquid when it is converted into vapor; the vapor-pressure curve being a curve of which abscissas represent boiling points (temperatures) and ordinates represent corresponding vapor pressures.

Equation (29) applies not only to vaporization but also to freezing. It is worth while, however, to discuss the case of freezing independently as follows. Consider

one gram of water initially in the form of ice at temperature T, the pressure and volume of the one gram of ice being p and V_i, respectively, as represented by the co-ordinates of the point a in Fig. 37. Imagine this one gram of water to be carried through a Carnot cycle as follows:

(1) Impart an amount of heat H_1 to the water (ice) until nearly all of the ice is melted at temperature T. This melting is accompanied by a great decrease of volume, and the process is represented by the line ab in Fig. 37.

(2) Increase the pressure by the amount Δp, without giving heat to or taking heat from the substance. This increase of pressure causes a decrease of the freezing point from T to $T-\Delta T$,* the infinitesimal residue of ice melts, and the heat absorbed

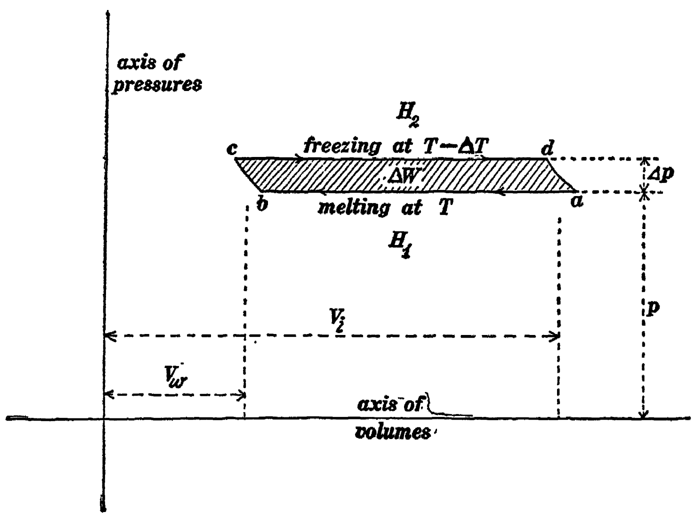


Fig. 37.

in this melting lowers the temperature of the whole to $T-\Delta T$. This process is represented by the curve bc in Fig. 37.

(3) Abstract an amount of heat H_2 from the substance until nearly all of the water is frozen at temperature $T-\Delta T$. This freezing is accompanied by a great increase of volume, and the process is represented by the curve cd in Fig. 37.

(4) Decrease the pressure to the original value without giving heat to or taking heat from the substance. This change of pressure causes the infinitesimal residue of water to be frozen, and the heat thus liberated raises the temperature of the substance from $T-\Delta T$ to T. This process is represented by the curve da in Fig. 37. The amount of heat H_1 is equal to the latent heat of fusion L of ice, and the amount of heat H_2 is equal to $H_1-\Delta W$, where ΔW is the work gained during the cycle;

*This statement that ΔT is a decrease of temperature must be kept in mind when one comes to interpret the final equation (30). It is more convenient to think of a decrease of temperature as being negative, and therefore the negative sign should be inserted in equation (30) giving equation (31).

and this work is represented by the shaded area in Fig. 37. Therefore from equation (15), we have

$$\frac{L}{L-(V_i-V_w)\cdot\Delta p}=\frac{H_1}{H_2}=\frac{T}{T-\Delta T}$$

or

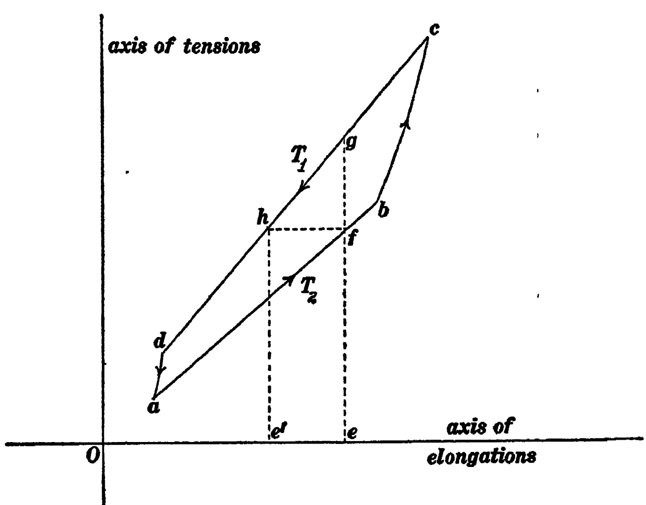
$$\frac{\Delta T}{\Delta p}=\frac{T}{L}(V_i-V_w) \tag{30}$$

in which ΔT is the decrease of freezing point due to an increase of pressure Δp. But ΔT is a decrease of temperature, and it is to be considered as negative so that equation (30) may be more properly written in the form

$$\frac{\Delta T}{\Delta p}=-\frac{T}{L}(V_i-V_w) \tag{31}$$

It is interesting to use this equation for calculating the decrease of freezing point of water due to an increase of pressure from one atmosphere to two atmospheres. Under these conditions, Δp = one atmosphere (equals, in round numbers, 1,000,000 dynes per square centimeter). T is equal to 273 degrees, L is equal to 80 calories (or $80 \times 4.2 \times 10^7$ ergs), V_w equals 1.000127 cubic centimeters, and V_i equals 1.090821 cubic centimeters; therefore equation (31) gives ΔT equal to 0.01338 centigrade degree.

63. Discussion of thermo-elastic properties of rubber. Take the ends of a rubber band in the hands, hold the band against the lips (which are very sensitive to changes of temperature), *stretch the band and notice that its temperature rises,*



correct diagram of rubber engine

Fig. 38.

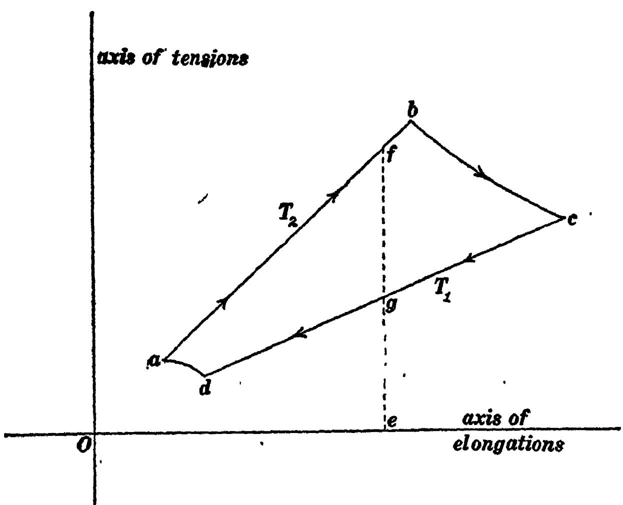
shorten the band and notice that its temperature falls. From these observed facts, it is immediately evident that *heat has to be abstracted from a rubber band to keep*

this temperature constant while it is being stretched, and that *heat has to be given to a rubber band to keep its temperature constant while it is being shortened.*

Starting from these facts, it can be shown by argument that *a rubber band which is stretched by hanging a given weight upon it is shortened when its temperature is raised slightly,* or, what amounts to the same thing, *if the band is prevented from shortening its tension increases when its temperature is raised slightly.* These results will be assumed as the basis of our argument and the argument will show that they are necessarily true.

When a rubber band is elongated, the tension of the band increases and a curve may be plotted of which the abscissas represent elongations and of which the ordinates represent tensions. Thus, the abscissas of the curve ab in Fig. 38 represent elongations and the corresponding ordinates represent tensions of the rubber band at constant temperature T_2.

Starting at any point a in Fig. 38 with the rubber band at temperature T_2, let the band be carried through a four-stage cycle as follows:



impossible diagram of rubber engine

Fig. 39.

(1) Increase the length of the band at constant temperature T_2 as represented by the line ab. During this process a definite amount of heat H_2 must be abstracted from the band.

(2) Continue to stretch the band, without abstracting heat from it, until its temperature rises to T_1. This part of the cycle is represented by the line bc in Fig. 38.

(3) Allow the band to shorten at constant temperature T_1 as represented by the line cd. During this process a definite amount of heat H_1 must be given to the band to keep its temperature constant.

(4) Allow the band to continue to shorten, without giving heat to it, until its temperature falls to T_2.

The net result of this cycle of operations is that a certain amount of heat H_1 is taken by the band from a region at high temperature T_1 and a certain amount of heat H_2 is delivered by the band to a region at low temperature T_2. But the operations (1), (2), (3) and (4) as above described are reversible, and reversible operations cannot bring about thermodynamic degeneration, therefore the transfer of a portion of the heat H_1 to a lower temperature must be compensated by a conversion of the remainder of this heat into work, that is to say, the work done by the band as it shortens at temperature T_1 must be greater than the work done on the band in stretching it at temperature T_2, or in other words, the tension *eg* must be greater than the tension *ef* as shown in Fig. 38. Given a rubber band under tension *ef* at temperature T_2, if the elongation remains unchanged, the tension must rise to *eg* when the temperature is raised from T_1 to T_2, or, what amounts

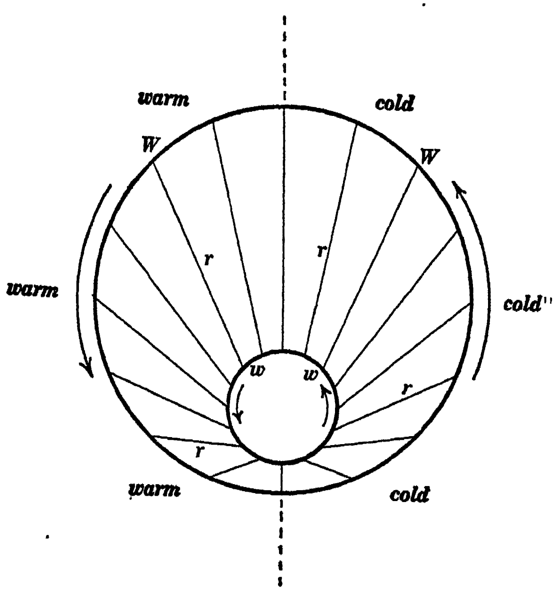


Fig. 40.

to the same thing, if the tension of the rubber band is kept constant, as, for example, by means of a weight hung upon it, then the increase of temperature from T_2 to T_1 must produce a shortening of the band from *Oe* to *Oe'*. If the tension of a rubber band were greater at the lower temperature T_2 than at the higher temperature T_1 as shown in Fig. 39, then to carry the band around the cycle *abcd* would result in (a) the doing of work on the band (the amount of work so done being converted into heat) and (b) the transfer of heat from the high temperature region T_1 to the low temperature region T_2 both of which effects would constitute thermodynamic degeneration. The process *abcd* is, however,

irreversible and cannot lead to thermodynamic degeneration. Therefore *eg* must be greater than *ef*.

Figure 40 shows the essential features of a rubber engine. The two wheels *WW* and *ww* lie in one plane and the axles of the wheels are geared together so that they must both rotate at the same speed in the same direction, bands of rubber *rrr* are stretched between the rims of the two wheels, one side of the whole apparatus is covered by a hood and kept warm, and the other side is exposed to the cool air. The greater tension of the warm rubber bands causes rotation in the direction of the curved arrows.*

Problems.

60. Pressure in Watt's diagram is represented by the ordinates to the scale 10 pounds per square inch of pressure per inch of ordinate, and volumes are represented by abscissas to the scale 100 cubic inches of volume per inch of abscissa. What amount of work in foot-pounds is represented by each square inch of area under a process curve? Ans. 83.3 foot-pounds.

61. Calculate the entropy in calories per degree centigrade of 10 grams of water in the form of saturated steam at 100°C. Base the calculation upon assumed constancy of specific heat of water from zero to 100° taking ice water as the zero state. Ans. 17.6 calories per degree centigrade.

Note. In solving this problem use algebraic integration or divide the rise of temperature from 0°C. to 100°C. into, say, 20 equal steps and calculate the quotient $\Delta H/T$ for each step.

62. A cylinder 12 inches long (in the clear) and 10 inches in diameter contains air at a pressure of 15 pounds per square inch. The air is compressed isothermally. Plot on cross section paper the curve showing the pressure for each position of the moving piston while the piston moves from a point 12 inches from the end of the cylinder to a point 6 inches from the end of the cylinder and calculate the work in foot-pounds done on the air during the compression. Ans. 815 foot-pounds.

Note. Calculate the work by determining the area under the plotted curve. To determine this area use the process of algebraic integration or divide the area up into a number of vertical strips and calculate the approximate area of each by considering the intervening portion of the curve as a straight line.

63. The gas in the above cylinder is compressed adiabatically. Plot the curve showing the pressure for each position of the piston in moving from the point 12 inches from the end of the cylinder to a point 6 inches from the end of the cylinder and calculate the work in foot-pounds done during the compression. Ans. 944 foot-pounds.

64. A perfect engine takes compressed air at 25°C. and 150 pounds per square inch pressure (above atmospheric pressure). At what cut-off must the engine be adjusted in order that the temperature of the expanded air may be −20°C.? Ans. 67.1 per cent.

*A series of interesting applications of the second law of thermodynamics is given in an important paper entitled "The Function of Osmotic Pressure in the Analogy between Solutions and Gases," by J. Van't Hoff, *Philosophical Magazine*, Vol. 26, pages 81–105, August, 1888.

Note. Assume the expansion in the cylinder to be adiabatic.

65. A bicycle pump is full of air at 15 pounds per square inch, length of stroke 15 inches. At what part of the stroke does air begin to enter the tire at 40 pounds per square inch above atmospheric pressure on the assumption that the compression takes place without rise of temperature? Ans. 4.113 inches from end of stroke.

66. Air at 15 pounds per square inch pressure and at 20°C. is pumped into a bicycle tire at a pressure of 40 pounds per square inch (above atmospheric pressure). Find the temperature of the compressed air, assuming the compression to be adiabatic, and find at what part of the stroke the air reaches 40 pounds per square inch above atmospheric pressure. Ans. 153°.5C.; 0.6 of stroke.

67. Air at 200 pounds per square inch (above atmospheric pressure) is used to drive an air engine which exhausts at 15 pounds per square inch (above atmospheric pressure). Required the temperature of the high pressure air in order that there may be no possibility of frost forming in the exhaust ports of the engine, the expansion of the air in the engine being assumed to be adiabatic. Ans. 212°C.

Note. Frost frequently forms in the exhaust ports of an air-driven engine. This occurs when the air (moist) is cooled below 0°C. by the expansion which takes place in the engine.

68. The atmospheric pressure at the ground is 755 millimeters. At a distance of 2,000 feet above the ground the pressure is 695 millimeters. The temperature of the air at the ground is 31°C. Find the value of the dew point of the air at the ground in order that a rising column of air would form a cloud at 2,000 feet above the ground. Ans. 25°.5 C.

Note. In this problem neglect the influence of the water vapor upon the law of adiabatic expansion of the air. Note that the pressure of the water vapor is decreased in the ratio 755 to 695 so that the air will have to be cooled below the temperature which expresses the dew point at the ground.

The following table showing relation between dew point and vapor pressure is needed in the solution of this problem.

Dew Point.	Vapor Pressure.
23°C.	20.9 mm.
24°C.	22.2 mm.
25°C.	23.5 mm.
26°C.	25.0 mm.

CHAPTER VII.

TRANSFER OF HEAT.

64. Conduction; convection; radiation. There are three quite distinct processes* by means of which heat is transferred from one place to another.

Conduction. If one end of a metal rod is held in a flame the whole rod becomes eventually more or less heated. Heat is transferred along the rod by being handed on from one part of the rod to the next part beyond. This mode of transfer of heat is called *heat conduction.* When one end of a copper rod is placed in a flame, the whole rod becomes heated in a very short time, heat spreads along an iron rod more slowly than along a copper rod, and the spread of heat along a glass rod, one end of which is held in a flame, is scarcely perceptible. Thus, copper is said to be a good heat conductor, iron not so good, and glass is a poor heat conductor. A very poor heat conductor is sometimes called a heat insulator. Porous substances such as saw-dust and wool are good heat insulators.

Convection. Heat may be transferred from one place to another by the flow of a hot fluid. Thus in a heating plant, heat flows from the furnace into the boiler by conduction and is then carried to the various parts of the building by hot water or steam. Great quantities of heat are carried by the winds and ocean currents from one place on the earth to another. This mode of heat transfer is called *heat convection* (see Art. 10).

Radiation. Heat is transferred from a hot to a cold substance through intervening space even when this space is entirely devoid of ordinary matter. This mode of transfer of heat is exemplified in the transfer of heat and light to the earth from the sun, and it is called *heat radiation.* The transfer of heat by radiation

*This term is not here used in the narrow sense defined in Art. 43.

shows itself by the burning sensation on one's face when one stands near an open fire when the air itself is cold. The transfer of heat by radiation is effected by wave motion like the wave motion which constitutes light, and these waves are transmitted by a medium, the ether, which fills all space. The molecular commotion in a hot body produces a commotion in the immediately adjacent ether, this commotion spreads out in all directions as a wave disturbance, and when these waves impinge on a cold body, they produce molecular commotion in it, thus heating it. This wave commotion in the ether is called radiant heat.*

Generally transfer of heat takes place by all three of the above processes simultaneously. Thus heat is distributed throughout a room from a hot stove partly by radiation (for one can feel with the hand the heat of the stove at a distance even though the air next the hand is quite cold), by currents of air (convection), and by conduction. The last process, however, is almost unappreciable in air inasmuch as air is a very poor heat conductor.

65. Temperature gradient. Fourier's law of heat conduction. The difference in temperature of a substance at two points divided by the distance apart of the points is called the *temperature gradient* between the points. Thus, one side of a wall is at 0°C., the other side is at 75°C., the wall is 25 centimeters thick, and the temperature gradient through the wall is 3 degrees per centimeter. Let t be the temperature of a substance at a point A, and let $t + \Delta t$ be the temperature of the substance at an adjacent point B distant Δx from A. Then $\Delta t/\Delta x$ is the temperature gradient between A and B.

The conductive flow of heat in a substance is always associated with a temperature gradient, the temperature of the substance falls off in the direction of the flow of heat or, in other words, heat flows from the warmer to the colder parts of the substance. Consider a metal rod of sectional area q along which heat is

*A fairly complete discussion of radiant heat is to be found in Franklin and MacNutt's *Light and Sound*, Appendix B, pages 301–316.

flowing by conduction. Let $\Delta t/\Delta x$ be the temperature gradient along the rod and let H be the number of units of heat flowing past a given section of the rod in a given time τ. Then H is proportional to q, to $\Delta t/\Delta x$, and to the time τ. We may, therefore write

$$H = Kq \cdot \frac{\Delta t}{\Delta x} \cdot \tau \tag{32}$$

in which K is a proportionality factor which is called the *thermal conductivity* of the given substance.* Thus, the thermal conductivity of copper at ordinary room temperature is about 0.95, that is to say, 0.95 calories of heat per second flow along a copper rod of which the sectional area is one square centimeter when the temperature gradient along the rod is one centigrade degree per centimeter. Expressed in terms of the same units the thermal conductivity of iron at ordinary room temperature is about 0.16; the thermal conductivity of glass at ordinary room temperature is about 0.0015; the thermal conductivity of paraffine at ordinary room temperature is about 0.0003; and the thermal conductivity of flannel cloth expressed in terms of the same units is about 0.000035.† The thermal conductivities of metals decrease with rise of temperature and the thermal conductivities of most other substances increase with rise of temperature.

66. Flow of heat through a wall. Consider a wall or slab of a substance of thickness d, the faces of the wall being at temperatures t' and t'' respectively. Let q be the sectional area of the wall (at right angles to the direction of flow of heat through the wall), and let K be the thermal conductivity of the substance of which the wall is made. Then $(t' - t'')/d$ is the temperature gradient through the wall, so that equation (32) gives

$$H = Kq\frac{(t' - t'')\tau}{d} \tag{33}$$

*Methods for experimentally determining thermal conductivity are discussed in Preston's *Theory of Heat*, pages 505–572 (London, Macmillan and Company, 1894).

†See Landolt and Bornstein's *Physikalisch-Chemische Tabellen* or Castell-Evans' *Physico-Chemical Tables*.

in which H is the number of units of heat flowing through the wall in τ seconds.

67. Emission* of heat. Curve of cooling. A hot body gives off heat to the surrounding air and to surrounding bodies by convection, by radiation, and, to a very slight extent, by conduction. The body is said to emit heat. The rate at which a body emits heat depends in a very complicated way upon the extent

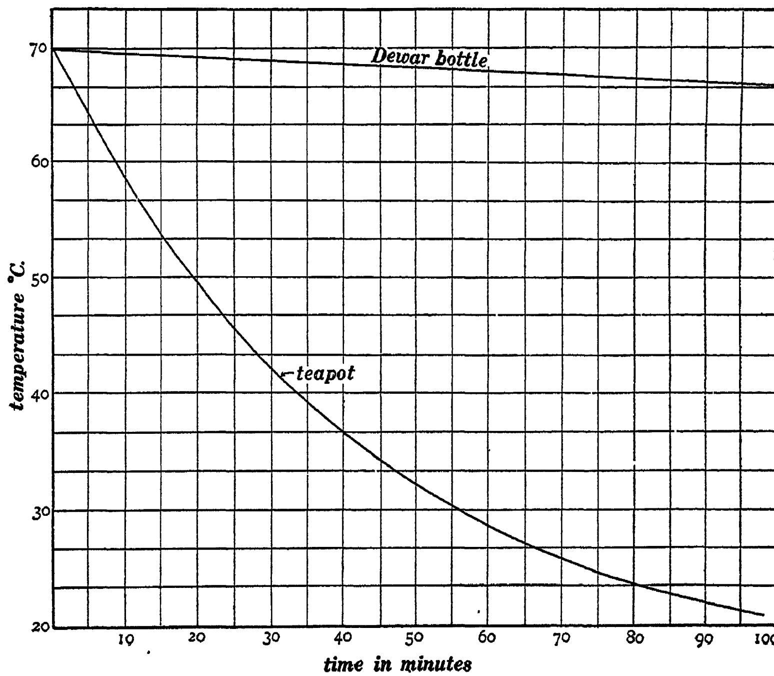


FIG. 41.

and character of the surface of the body, upon the nature of the surrounding gas and its freedom of motion, and upon the nature of surrounding bodies. For any given case a body may be heated and then as it cools its temperature may be observed at intervals,

*This term as used in treatises on the theory of radiation refers to the emission of heat by a body by radiation only. Here the term means the giving off of heat by a body to the surrounding air and to surrounding bodies by conduction, by convection, and by radiation.

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and the result of such a series of observations may be plotted in the form of a curve which is called the cooling curve of the given body under the given conditions. Thus, Fig. 41 shows the cooling curve of an ordinary teapot filled at the start with hot water and allowed to stand on a table. The figure also shows the first part of the cooling curve of a Dewar bottle* containing hot water.

PROBLEMS.

69. A boiler shell is made of iron one centimeter thick, and 20,000 calories per hour flow through each square centimeter of the shell. What is the temperature difference between inner and outer surfaces of shell? What would this temperature difference be with a copper shell of the same thickness? Use values of thermal conductivity given on page 400. Ans. 34°.7 C.; 5°.78 C.

70. The inside surface of the window-glass in a house is at a temperature of 15°C., the outside surface is at a temperature of − 10°C., and the glass is 4 millimeters thick. Calculate the heat in calories which during 24 hours flows out of the house by conduction through a total window-glass area of 20 square meters and reduce the result to kilograms of coal at 6000 calories per gram. Use the value of thermal conductivity given on page 400. Ans. 1728×10^6 calories; 288 kilograms of coal.

71. The thermal conductivity of iron is 0.16 when heat flow is expressed in calories per second per square centimeter and temperature gradient in centigrade degrees per centimeter. What is the thermal conductivity of iron when heat flow is expressed in British thermal units per square inch per second and temperature gradient in Fahrenheit degrees per inch?

72. A metal vessel containing 25,200 grams of water has a flat face 30 × 30 centimeters which is pressed against the outside of

*The Dewar bottle is a double-walled glass bottle with the air exhausted from the space between the walls. The inner surfaces of the walls are usually silvered. The Dewar bottle is sold under various trade names such as the "Thermos" bottle or the "Hotakold" bottle. The remarkable property of the Dewar bottle is explained in Franklin and MacNutt's *Light and Sound* Appendix B.

a furnace wall made of brick, and the temperature of the water is observed to rise 7°.5C. in twenty minutes. The wall is 30 centimeters thick, the inner face of the wall is at a temperature of 1500°C. and the outer face of the wall is at 25°C. What is the thermal conductivity of the material of which the wall is made? Ans. 0.00356.

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